REDUCTIONS OF PIECEWISE TRIVIAL PRINCIPAL COMODULE ALGEBRAS

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ABSTRACT. Let G' be a closed subgroup of a topological group G. A principal G-bundle X is reducible to a locally trivial principal G'-bundle X' if and only if there exists a local trivialisation of X such that all transition functions take values in G'. We prove a noncommutative-geometric counterpart of this theorem. To this end, we employ the concept of a piecewise trivial principal comodule algebra as a suitable replacement of a locally trivial compact principal bundle. To illustrate our theorem, first we define a noncommutative deformation of the $\mathbb{Z}/2\mathbb{Z}$-principal bundle $S^2 \to \mathbb{R}P^2$ that yields a piecewise trivial principal comodule algebra. It is the C*-algebra of a quantum cube whose each face is given by the Toeplitz algebra. The $\mathbb{Z}/2\mathbb{Z}$-invariant subalgebra defines the C*-algebra of a quantum $\mathbb{R}P^2$. It is given as a triple-pullback of Toeplitz algebras. Next, we prolongate this noncommutative $\mathbb{Z}/2\mathbb{Z}$-principal bundle to a noncommutative $U(1)$-principal bundle, so that the former becomes a reduction of the latter instantiating our theorem. Moreover, using K-theory results, we prove that the prolongated noncommutative bundle is not trivial.

CONTENTS

Introduction and preliminaries

The aim of this article is to provide a critirion for a reducibility of piecewise trivial comodule algebras. More precisely, given a Hopf algebra H with bijective antipode, an appropriate Hopf ideal J, and a principal H-comodule algebra P, we claim that:

THEOREM *There exists an ideal $I \subseteq P$ such that P/I is a piecewise trivial principal H/J-comodule algebra if and only if there exists a piecewise trivialisation of P (with respect to the same covering) such that all the associated transition functions anihilate J and its associated action on the algebras covering the subalgebra of coaction invariants is trivial.*

Our main tool in proving this result is the Hopf-Galois Reduction Theorem [21, 11, 16] establishing the equivalence of reduction ideals I and appropriate equivariant algebra homomorphism. The latter have a geometric meaning of global sections of the fibre bundle associated to a principal G-bundle via the canonical action $G \times G/G' \to G/G'$, where G' is a reducing subgroup of G. They turn out to be far more manageable than reduction ideals.

We work over a fixed ground field k. The unadorned tensor product stands for the tensor product over this filed. The comultiplication, counit and the antipode of a Hopf algebra H are denoted by Δ, ε and S, respectively. Our standing assumption is that S is invertible. A right H-comodule algebra P is a unital associative algebra equipped with an H-coaction $\Delta_P : P \to P \otimes H$ that is an algebra map. For a comodule algebra P, we call

$$(0.1) \qquad P^{\operatorname{co}H} := \{p \in P \,|\, \Delta_P(p) = p \otimes 1\}$$

the subalgebra of coaction-invariant elements in P. A left coaction on V is denoted by ${}_V\Delta$. For comultiplications and coactions, we often employ the Heynemann-Sweedler notation with the summation symbol suppressed:

$$(0.2) \qquad \Delta(h) =: h_{(1)} \otimes h_{(2)}, \quad \Delta_P(p) =: p_{(0)} \otimes p_{(1)}, \quad {}_V\Delta(v) =: v_{(-1)} \otimes v_{(0)}.$$

The convolution product of f and g is denoted by

$$(0.3) \qquad (f * g)(h) := f(h_{(1)})g(h_{(2)}).$$

Finally, we use the convention that ${}_A^C\operatorname{Hom}_B^D$ signifies k-linear homomorphisms that are left A-linear, right B-linear, left C-colinear and right D-colinear. If M is a right comodule over a coalgebra C and N is a left C-comodule, then we define their *cotensor product* as

$$(0.4) \qquad M\,{\overset{C}{\square}}\,N := \{t \in M \otimes N \,|\, (\Delta_M \otimes \operatorname{id})(t) = (\operatorname{id} \otimes\, {}_N\Delta)(t)\}.$$

In particular, for a right H-comodula algebra P and a left H-comodule V, we observe that $P\square^H V$ is a left $P^{\operatorname{co}H}$-module in a natural way.

0.1. Reductions, prolongations and local triviality of classical principal bundles.
Let $\pi : X \to M$ be a principal G-bundle over M, and G' a subgroup of G. A *G'-reduction* of $X \to M$ is a sub-bundle $X' \subseteq X$ over M that is a principal G'-bundle over M via the restriction of the G-action on X. The concept of a reduction is crucial because many important structures on manifolds can be formulated as reductions of their frame bundles. For instance, an orientation,

a volume form and a metric on a manifold M correspond to reductions of the frame bundle FM to a $GL_+(n, \mathbb{R})$-, $SL(n, \mathbb{R})$- and $O(n, \mathbb{R})$-bundle, respectively. See [20] for more details.

LEMMA 0.1. *Let G' be a closed subgroup of G. Suppose that a principal G-bundle X is reducible to a principal G'-bundle X'. Then*

$$(0.5) \qquad X \ni x \longmapsto [x', g] \in X' \times_{G'} G, \quad where \ x'g = x,$$

$$(0.6) \qquad X' \times_{G'} G \ni [x', g] \longmapsto x'g$$

is a pair of mutually inverse gauge isomorphisms.

THEOREM 0.2. *(cf. [19]) Let G' be a closed subgroup of G. A principal G-bundle X is reducible to a principal G'-bundle X' if and only if there exists a right G map $f : X \to G' \backslash G$. Explicitly, given map f, the reduced subbundle can be recovered as $X' = f^{-1}([e])$. On the other hand, having a G' reduction X' we can construct appropriate f by composing the isomorphism 0.5 with projection on the second component and quotient map:*

$$x \mapsto [(x', g)] \mapsto {}_{G'}[g]$$

LEMMA 0.3. *A principal G-bundle X is isomorphic as a G-space with $X/G \times G$ if and only if there exists a right G-map $\Phi : X \to G$. Then the isomorphism is given explicitly by*

$$(0.7) \qquad X \ni x \longmapsto ([x], \Phi(x)) \in X/G \times G, \quad X/G \times G \ni ([x], g) \longmapsto x\Phi(x)^{-1}g \in X.$$

Note that Ehresmann groupoid $G \times_{G'} G$ which can be thought of as G-prolongation of G treated as principal G'-bundle is trivial as a G-bundle, due to the above lemma. Indeed the map $\Phi : G \times_{G'} G \to G$ is given here by multiplication:

$$(0.8) \qquad [g, h] \longmapsto gh.$$

The reducibility of a locally trivial principal bundle can be phrased in terms of transition functions (cf. [20], Proposition I.5.3):

PROPOSITION 0.4. *Let G' be a closed subgroup of G. A principal G-bundle $\pi : X \to M$ is reducible to a locally trivial principal G'-bundle X' if and only if there exists a local trivialisation of X (with respect to the same covering as that of X') such that all transition functions take values in G'.*

In particular, the structure groups of trivial bundles can be reduced to arbitrary subgroups.

Note that a reduction of a trivial bundle need not be trivial.

As an example let us consider the boundary of the Möbius strip is a nontrivial $\mathbb{Z}/2\mathbb{Z}$-bundle over S^1 that can be obtained as a reduction of the trivial $U(1)$-bundle over S^1.

According to Proposition 0.2 the reductions of $S^1 \times U(1)$ are in one to one correspondence with right $U(1)$ maps $f : S(1) \times U(1) \to (\mathbb{Z}/2\mathbb{Z}) \backslash U(1)$. Let us consider two choices of such maps:

$$(0.9) \qquad f_1 : S^1 \times U(1) \ni (s, u) \longmapsto [su] \in (\mathbb{Z}/2\mathbb{Z}) \backslash U(1),$$

$$(0.10) \qquad f_2 : S^1 \times U(1) \ni (s, u) \longmapsto [s^{1/2}u] \in (\mathbb{Z}/2\mathbb{Z}) \backslash U(1).$$

It is easy to see that $f_1^{-1}([e]) \simeq S^1 \times \mathbb{Z}/2\mathbb{Z}$. Explicitly, $f_1^{-1}([e]) = \{(\pm u, u^{-1}) \mid u \in U(1)\}$, where we identify s^1 with $U(1)$. Note that the action of $\mathbb{Z}/2\mathbb{Z}$ on $f_1^{-1}([e])$ sends an element of one circle (u, u^{-1}) to the element $(u, -u^{-1}) = (-(-u), (-u)^{-1})$ which belongs to the other circle.

On the other hand, $(s, u) \in f_2^{-1}([e])$ if and only if $s^{-1/2}u = \pm e$, i.e., $s = u^2$, hence $f_2^{-1}([e])$ is isomorphic with S^1, the explicit isomorphism given by $u \mapsto (u^2, u)$. Note that the action of $\mathbb{Z}/2\mathbb{Z}$ sends parameter u to $-u$. It is easy to see that S^1 with this action is an edge of Möbius strip.

Therefore, one has to bear in mind that a local trivialisation of a principal G-bundle X when restricted to a reduced G'-sub-bundle X' need not be a trivialisation of X'. The clue is that the principal bundle $U(1) \to U(1)/(\mathbb{Z}/2\mathbb{Z})$ is not trivial. Its triviality would be a sufficient condition for the triviality of the reduction:

PROPOSITION 0.5. *If $G \to G/G'$ is trivial as G'-bundle, then any G'-reduction of a trivial G-bundle is trivial.*

Finally, recall that reductions of principal bundles are classified by the global sections of appropriate associated fibre bundles [19, Theorem 2.3]. More precisely, a G-principal bundle $X \to M$ can be reduced to a G'-sub-bundle if and only if there exists a global section of the associated fibre bundle $\pi : X/G' \to M$. There is a natural way to provide a one-to-one correspondence between the G'-reductions of X and global sections of X/G'. It supports the geometric intuition of a G'-sub-bundle as a G'-thick global section of X. The group inverse allows us to identify G/G' with $G'\backslash G$ and G-equivariant maps into G/G' with G-equivariant maps into $G'\backslash G$: $f : X \to G'\backslash G$, $f(xg) = f(x)g$. Finding a noncommutative counterpart of these maps is the backbone of the Hopf-Galois Reduction Theorem.

THEOREM 0.6. *Let G' be a closed subgroup of a topological group G. A principal G-bundle X is reducible to a locally trivial principal G'-bundle X' if and only if there exists a local trivialisation of X such that all transition functions take values in G'.*

0.2. **Reductions and prolongations of principal comodule algebras.**

Let H be a Hopf algebra, P be a right H-comodule algebra and let $B := P^{\mathrm{co}\,H}$ be the coaction-invariant subalgebra. The H-comodule algebra P is called a *principal* [6] if:

(1) $P\otimes_B P \ni p \otimes q \mapsto \mathrm{can}(p \otimes q) := pq_{(0)} \otimes q_{(1)} \in P \otimes H$ is bijective,
(2) $\exists s \in {}_B\mathrm{Hom}^H(P, B \otimes P) : m \circ s = \mathrm{id}$, where m is the multiplication map,
(3) the antipode of H is bijective.

Here (1) is the Hopf-Galois (freeness) condition, (2) means equivariant projectivity of P, and (3) ensures a left-right symmetry of the definition (everything can be re-written for left comodule algebras). The inverse of can can be written explicitly using Heynemann-Sweedler like notation: $\mathrm{can}^{-1}(p \otimes h) := ph^{[1]} \otimes_B h^{[2]}$. Here the map

$$(0.11) \qquad\qquad H \ni h \longmapsto \mathrm{can}^{-1}(1 \otimes h) =: h^{[1]} \underset{B}{\otimes} h^{[2]} \in P \underset{B}{\otimes} P$$

is called a *translation map*. It enjoys the following property which we will use later on:

$$(0.12) \qquad\qquad h^{[1]}h^{[2]} = \varepsilon(h).$$

If H is a Hopf algebra with bijective antipode and P is a right H-comodule algebra, then one can show (cf. [6]) that it is principal if and only if there exists a linear map

$$(0.13) \qquad \ell : H \longrightarrow P \otimes P, \quad h \longmapsto \ell(h) =: \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle},$$

that, for all $h \in H$, satisfies:

$$(0.14) \qquad\qquad \ell(h)^{\langle 1 \rangle} \ell(h)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}{}_{(1)} = 1 \otimes h,$$

$$(0.15) \qquad S(h_{(1)}) \otimes \ell(h_{(2)})^{\langle 1 \rangle} \otimes \ell(h_{(2)})^{\langle 2 \rangle} = \ell(h)^{\langle 1 \rangle}{}_{(1)} \otimes \ell(h)^{\langle 1 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle},$$

$$(0.16) \qquad \ell(h_{(1)})^{\langle 1 \rangle} \otimes \ell(h_{(1)})^{\langle 2 \rangle} \otimes h_{(2)} = \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}{}_{(1)}.$$

Any such a map ℓ can be made unital [6]. It is then called a *strong connection* [12, 10, 6], and can be thought of as an apropriate lifting of the translation map.

Let $H \ni h \mapsto \ell(h) = \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle} \in P \otimes P$ be a strong connection on P, and the map

$$(0.17) \qquad s : P \ni p \longmapsto p_{(0)} \ell(p_{(1)})^{\langle 1 \rangle} \otimes \ell(p_{(1)})^{\langle 2 \rangle} \in B \otimes P$$

its associated splitting of the multiplication map.

A particular class of principal comodule algebras is distinguished by the existence of a cleaving map. A cleaving map is defined as a unital right H-colinear convolution-invertible map $j : H \to P$. Having a cleaving map, one can define a strong connection as

$$(0.18) \qquad\qquad \ell := (j^{-1} \otimes j) \circ \Delta,$$

where j^{-1} stands for the convolution inverse of j. Comodule algebras admitting a cleaving map are called *cleft*. All modules associated with cleft comodule algebras are always free. Also, one can show that a cleaving map is automatically injective. Therefore, as the value of a cleaving map on a group-like element is invertible, we can conclude that the existence of a non-trivial group-like in H necessitates the existence of an invertible element in P that is not a multiple of 1. Hence one of the ways to prove the non-cleftness of a principal comodule algebra over a Hopf algebra with a non-trivial group-like is to show the lack of non-trivial invertibles in the comodule algebra.

If $j : H \to P$ is a right H-colinear algebra homomorphism, then it is automatically convolution-invertible and unital. A cleft comodule algebra admitting a cleaving map that is an algebra homomorphism is called a *smash product*. All commutative smash products reduce to the tensor algebra $P^{\mathrm{co}\,H} \otimes H$, so that smash products play the role of trivial bundles. A cleaving map defines a left action of H on $P^{\mathrm{co}\,H}$ making it a left H-module algebra: $h \triangleright p := j(h_{(1)})p j^{-1}(h_{(2)})$. Conversely, if B is a left H-module algebra, one can construct the smash product $B \rtimes H$ by equipping the vector space $B \otimes H$ with the multiplication

$$(0.19) \qquad (a \otimes h)(b \otimes k) := a\,(h_{(1)} \triangleright b) \otimes h_{(2)}\,k, \quad a, b \in B, \ h, k \in H,$$

and coaction $\Delta_{B \rtimes H} := \mathrm{id} \otimes \Delta$. Then a cleaving map is simply given by $j(h) = 1 \otimes h$. Plugging it into the formula (0.18) yields a strong connection defined by

$$(0.20) \qquad \ell : H \longrightarrow (B \rtimes H) \otimes (B \rtimes H), \quad h \longmapsto (1 \otimes S(h_{(1)})) \otimes (1 \otimes h_{(2)}).$$

LEMMA 0.7. *Let* $A \xrightarrow{\delta} A \otimes k\Gamma$ *be a Galois co-action, and*

$$A_\gamma := \{a \in A \mid \delta(a) = a \otimes \gamma\}.$$

Then $\{e_i\}_{i=1}^n$ *is a basis of* A_γ *if and only if there exists* $\{f_i\}_{i=1}^n \subseteq A_{\gamma^{-1}}$ *such that*

$$(0.21a) \qquad \forall i, j \in \{1, \ldots, n\} : e_j f_i = \delta ij,$$

$$(0.21b) \qquad \sum_{i=1}^n f_i e_i = 1.$$

LEMMA 0.8. *Let* P *be a principal co-module algebra over* H, *and* $P \xrightarrow{\chi} k$ *be a character. The for any left co-module* V, *we have* $k \otimes_{P^{\mathrm{co}\,H}} (P \square_H V) = V$.

Proof. Trivially, $k \otimes_{P^{\mathrm{co}\,H}} (P \square_H V) = k \otimes_P (P \otimes_{P^{\mathrm{co}\,H}} (P \square_H V))$. Then, by the right flatness of P implied by the principality this equals to $k \otimes_P ((P \otimes_{P^{\mathrm{co}\,H}} P) \square_H V) = k \otimes_P ((P \otimes H) \square_H V) = k \otimes_P (P \otimes (H \square_H V)) = k \otimes_P (P \otimes V) = (k \otimes_P P) \otimes V = k \otimes V = V$. Note that we have used the right exactness: $0 \to 0 \to M \to N \to 0$ is exact implies that $F \otimes_R 0 \to F \otimes_R M \to F \otimes_R N \to 0$ is exact. $\qquad \square$

COROLLARY 0.9. *If, in addition, there exists a character* $A \xrightarrow{\chi} k$, *then* A_γ *is free if and only if there exists* $x \in A_\gamma$ *and* $x^{-1} \in A_{\gamma^{-1}}$.

Proof. Assume first that A_γ is free. Then by Lemma 0.8 there exists $x \in A_\gamma$ such that $\{x\}$ $\{x\}$ is a basis of A_γ. This sets $n = 1$ in the foregoing Lemma. $\qquad \square$

DEFINITION 0.10 ([11, 21, 16]). *Let* P *be a principal* H-*comodule algebra with* $B = P^{\mathrm{co}\,H}$ *and* J *be a Hopf ideal of* H *such that* H *is a principal left* H/J-*comodule algebra. We say that an ideal* I *of* P *is a* J-*reduction of* P *if and only if the following conditions are satisfied:*

(1) *I is an H/J-subcomodule of P,*
(2) *P/I with the induced coaction is a principal H/J-comodule algebra,*
(3) *$(P/I)^{\mathrm{co}\,H/J} = B$.*

Losely speaking, J plays the role of the ideal of functions vanishing on a subgroup and I the ideal of functions vanishing on a sub-bundle. Thus H/J works as the algebra of the reducing subgroup and P/I the algebra of the reduced bundle. The coaction invariant subalgebra B remains intact — the base space of a sub-bundle coincides with the base space of the bundle.

The space of all such J-reducing ideals we denote by ${}_B\mathrm{Red}^{H/J}(P)$. This set can be empty, as for a given J there need not exist a reduction. If no non-zero J admits a reduction, we say that the extension is *irreducible*. The thus defined reductions have clear conceptual meaning but are difficult to handle. Following the classical case (see Introduction), one can prove that

they are equivalent to right H-colinear algebra homomorphisms from the left coaction invariant subalgebra ${}^{coH/J}H$ to the centralizer subalgebra $Z_P(B) := \{p \in P \mid pb = bp, \ \forall \, b \in B\}$ that are compatible with the Miyashita-Ulbrich action. The latter condition (trivial in the commutative case) means that

$$(0.22) \qquad f(S(h_{(1)})kh_{(2)}) = h^{[1]}f(k)h^{[2]}, \ \forall \, k \in {}^{coH/J}H, \ h \in H.$$

The space of all such homomorphisms we denote by $\mathrm{Alg}_H^H({}^{coH/J}H, Z_P(B))$. Note that $S(h_{(1)})kh_{(2)} \in {}^{coH/J}H$ for all $k \in {}^{coH/J}H$, $h \in H$.

THEOREM 0.11 (Hopf-Galois Reduction [11, 21, 16]). *Let P be a principal H-comodule algebra, and $B := P^{co\,H}$. Then the formulas*

$$(0.23) \qquad \mathrm{Alg}_H^H({}^{coH/J}H, Z_P(B)) \ni f \ \longmapsto \ I_f := Pf({}^{coH/J}H \cap \mathrm{Ker}\,\varepsilon) \in {}_B\mathrm{Red}^{H/J}(P),$$

$${}_B\mathrm{Red}^{H/J}(P) \ni I \ \longmapsto \ f_I \in \mathrm{Alg}_H^H({}^{coH/J}H, Z_P(B)),$$

$$(0.24) \qquad f_I(k) \quad := \quad S^{-1}(k)^{[1]}(i_B \circ \pi_I)(S^{-1}(k)^{[2]}),$$

$$i_B(\pi_I(b + x)) \quad := \quad b, \quad i_B : (B \oplus I)/I \to B, \quad b \in B, \quad x \in I,$$

define mutually inverse bijections.

1. REDUCTIONS OF PIECEWISE TRIVIAL COMODULE ALGEBRAS

1.1. Piecewise triviality revisited.
A family of surjective algebra morphisms $\{\pi_i : P \to P_i\}_{i \in \{1,\ldots,N\}}$ is called a *covering* [15] when

(1) $\bigcap_{i \in \{1,\ldots,N\}} \mathrm{Ker}\,\pi_i = \{0\}$,
(2) The family of ideals $(\mathrm{Ker}\,\pi_i)_{i \in \{1,\ldots,N\}}$ generates a distributive lattice with $+$ and $\cap$ as meet and join respectively.

Let $\{\pi_i : P \to P_i\}_i$ be a covering. We define the family of canonical surjections

$$(1.1) \qquad \pi_j^i : P_i \to P/(\mathrm{Ker}\,\pi_i + \mathrm{Ker}\,\pi_j), \quad \pi_i(p) \mapsto p + \mathrm{Ker}\,\pi_i + \mathrm{Ker}\,\pi_j,$$

and denote by P^c the multipullback of P_i's along π_j^i's:

$$(1.2) \qquad P^c := \{(p_i)_i \in \Pi_i P_i \mid \pi_j^i(p_i) = \pi_i^j(p_j)\}.$$

The following Proposition states the relationship between P and P^c.

PROPOSITION 1.1 ([8]). *Let $\{\pi_i : P \to P_i\}_{i \in \{1,\ldots,N\}}$ be a covering. Then the map*

$$(1.3) \qquad \chi : P \longrightarrow P^c, \quad p \longmapsto (\pi_i(p))_i$$

is an algebra isomorphism. (If P and all the P_i's are H-comodule algebras for some Hopf algebra H and all the π_i's are colinear, then so is χ.)

The isomorphism (1.3) is what makes the notion of the covering so much useful, as it often allows to glue the properties of the parts of P (the P_i's) into the properties of the whole of P.

We recall now the notion of a quantum version of piecewise triviality of the bundle (like local triviality, but with respect to closed subsets):

DEFINITION 1.2 ([15]). *An H-comodule algebra P is called piecewise trivial if there exists a family of surjective $\{\pi_i : P \to P_i\}_{i \in \{1,\dots,N\}}$ H-colinear maps such that:*

(1) *the restrictions $\pi_i|_{P^{\mathrm{co}\,H}} : P^{\mathrm{co}\,H} \to P_i^{\mathrm{co}\,H}$ form a covering,*
(2) *the P_i's are smash products ($P_i \cong P_i^{\mathrm{co}\,H} \rtimes H$ as H comodule algebras).*

Note that, if the antipode of H is bijective, then it follows from the main result of [15] that P is principal – this is an important instance of gluing of properties mentioned above. To emphasize this fact and stay in touch with the classical terminology, we frequently use the phrase "piecewise trivial principal comodule algebra".

Note also that the consequence of principality of P is that $\{\pi_i : P \to P_i\}_{i \in \{1,\dots,N\}}$ is a covering of P. To see that one can use [15, Proposition 3.4] which states that $K \mapsto K \cap P^{\mathrm{co}\,H}$ is a lattice monomorphism between the lattice of ideals in P which are right H-comodules and the lattice of ideals in $P^{\mathrm{co}\,H}$. Indeed, we have that $P^{\mathrm{co}\,H} \cap \bigcap_i \mathrm{Ker}\,\pi_i = \bigcap_i (\mathrm{Ker}\,\pi_i \cap P^{\mathrm{co}\,H}) = 0$ by assumption, and so $\bigcap_i \mathrm{Ker}\,\pi = 0$ by the injectivity of $P^{\mathrm{co}\,H} \cap \cdot$. Similarly, the distributivity follows as $P^{\mathrm{co}\,H} \cap \cdot$ maps monomorphically the lattice generated by $\mathrm{Ker}\,\pi_i's$ into a distributive lattice.

The following Lemma is the slight generalization of the result implicit in the proof of [15, Proposition 3.4]. It is used in the proof of our main result, but it is also interesting on its own.

LEMMA 1.3. *Let P be a principal H-comodule algebra and $B = P^{\mathrm{co}\,H}$. Let K be an ideal and a right H-subcomodule of P, and let L be an ideal in B. Then $L = K \cap B$ if and only if $K = LP$.*

Proof. Assume first that $K = LP$. It is obvious that $L \subseteq B \cap K$. To prove the converse inclusion, take any $p := \sum_i l_i p_i \in K \cap B$, where $l_i \in L$, $p_i \in P$, for all i. Taking advantage of the splitting (0.17) provided by a strong connection and any unital linear functional f on P, we compute

$$(1.4) \qquad p = p\ell(1)^{\langle 1 \rangle} f(\ell(1)^{\langle 2 \rangle}) = p_{(0)}\ell(p_{(1)})^{\langle 1 \rangle} f(\ell(p_{(1)})^{\langle 2 \rangle}) = \sum_i l_i p_{i(0)}\ell(p_{i(1)})^{\langle 1 \rangle} f(\ell(p_{i(1)})^{\langle 2 \rangle}).$$

Hence $p \in L$ as $p_{i(0)}\ell(p_{i(1)})^{\langle 1 \rangle} f(\ell(p_{i(1)})^{\langle 2 \rangle}) \in B$ and L is an ideal in B.

Conversely, assume that $L = B \cap K$. The inclusion $LP \subseteq K$ is obvious because K is an ideal in P. To show the opposite inclusion, apply the splitting (0.17) to any $p \in K$. Then

$$(1.5) \qquad\qquad B \otimes P \ni p_{(0)}\ell(p_{(1)})^{\langle 1 \rangle} \otimes \ell(p_{(1)})^{\langle 2 \rangle} \in K \otimes P$$

because K is a subcomodule and an ideal in P. Hence

$$(1.6) \qquad\qquad p = p_{(0)}\varepsilon(p_{(1)}) = p_{(0)}\ell(p_{(1)})^{\langle 1 \rangle}\ell(p_{(1)})^{\langle 2 \rangle} \in (B \cap K)P = LP,$$

as needed. $\square$

Finally we recall quantum versions of the the concepts of a piecewise trivialisation and transition functions:

DEFINITION 1.4. *Let $\{\pi_i : P \to P_i\}_i$ be a covering by right H-colinear maps of a principal right H-comodule algebra P such that the restrictions $\pi_i|_{P^{\mathrm{co}\,H}} : P^{\mathrm{co}\,H} \to P_i^{\mathrm{co}\,H}$ also form a*

covering. A piecewise trivialisation *of P with respect to the covering* $\{\pi_i : P \to P_i\}_i$ *is a family* $\{\gamma_i : H \to P_i\}_i$ *of right H-colinear algebra homomorphisms (cleaving maps).*

It is clear that a principal comodule algebra is piecewise trivial if and only if it admits a piecewise trivialisation. With each piecewise trivialisation of P we can associate the *transition functions*

$$(1.7) \qquad T_{ij} := (\pi_j^i \circ \gamma_i) * (\pi_i^j \circ \gamma_j \circ S) : \ H \longrightarrow P/(\operatorname{Ker} \pi_i + \operatorname{Ker} \pi_j),$$

where π_j^i's are given by (1.1). It follows directly from the colinearity of π_j^i's and γ_j's that the elements in the images of all the T_{ij}'s are coaction invariant. Combining this with the fact that intersecting kernels of π_j's with coaction invariant subalgebra defines a homomorphism of lattices [15, Proposition 3.4], we conclude that the image of each T_{ij} is contained in $P^{\operatorname{co} H}/(\operatorname{Ker} \pi_i|_{P^{\operatorname{co} H}} + \operatorname{Ker} \pi_j|_{P^{\operatorname{co} H}})$.

As in the classical setting, transition functions can be used to assemble a principal comodule algebra from trivial pieces. Indeed, (1.2) can be rewritten as

$$(1.8) \ \ P^c = \{(p_i)_i \in \prod_i P_i \mid \pi_j^i(p_{i(0)}\gamma_i(S(p_{i(1)})))T_{ij}(p_{i(2)}) \otimes p_{i(3)} = \pi_i^j(p_{j(0)}\gamma_j(S(p_{j(1)}))) \otimes p_{j(2)}\}.$$

Since for any i and j we have $\operatorname{Im} T_{ij} \subseteq P^{\operatorname{co} H}/(\operatorname{Ker} \pi_i|_{P^{\operatorname{co} H}} + \operatorname{Ker} \pi_j|_{P^{\operatorname{co} H}})$ and $p_{(0)}\gamma_i(S(p_{(1)})) \in P_i^{\operatorname{co} H}$, the compatibility conditions defining P^c all take place at the base-space (coaction invariant) algebras.

We are now ready to state the main result of this paper:

THEOREM 1.5. *Let P be a principal right H-comodule algebra, and J a Hopf ideal of H such that H is a principal left H/J-comodule algebra. Then there exists a J-reduction of P to a piecewise trivial principal right H/J-comodule algebra if and only if there exists a piecewise trivialisation of P (with respect to the same covering $\{B \to B_i\}_{i \in \{1, \cdots, N\}}$ as that of the J-reduction) such that $T_{ij}(J) = 0$ for all the associated transition functions T_{ij} and $J \triangleright_i B_i = 0$ for all the actions $H \otimes B_i \to B_i$, $h \triangleright_i b := \gamma_i(h_{(1)})b\gamma_i(S(h_{(2)}))$.*

1.2. **A proof of the main theorem.** Our proof consists of two parts each of which establishes one of the implications of the asserted equivalence, and both parts are divided into several lemmata. First we provide lemmata needed for proving the implication "the existence of a trivialisation with some properties implies that there exists a reduction to a piecewise trivial comodule algebra".

Our first lemma is a certain general statement needed in the second lemma.

LEMMA 1.6 ([16]). *Let L be a bialgebra and $\overline{L}$ be a coalgebra and a left L-module. Assume that there exists a surjective left L-linear coalgebra map $\pi : L \to \overline{L}$, and view L as a left $\overline{L}$-comodule with the coaction $_L\Delta = (\pi \otimes \operatorname{id}) \circ \Delta$. Then*

$$(1.9) \qquad\qquad D := {}^{\operatorname{co}\overline{L}}L = \{d \in L \mid {}_L\Delta(d) = \pi(1) \otimes d\}$$

is a right L-comodule subalgebra of L, i.e. $\Delta(D) \subseteq D \otimes L$. Furthermore, the augmentation ideal $D^+ := D \cap \operatorname{Ker} \varepsilon$ is contained in $\operatorname{Ker} \pi$ and $\forall\, d \in D : \Delta(d) - 1 \otimes d \in D^+ \otimes L$.

In the following lemma we prove the existence of a reduction of a trivial (smash product) comodule algebra when the trivialising map satisfies certain condition.

LEMMA 1.7. *Let P be a smash product H-comodule algebra, $B := P^{\mathrm{co}\,H}$, and $\gamma : H \to P$ a cleaving map. Let J be a Hopf ideal of H such that $h \triangleright b := \gamma(h_{(1)})b\gamma(S(h_{(2)})) = 0$ for all $h \in J$ and $b \in B$. Then γ restricts to an element of $\mathrm{Alg}_H^H(^{\mathrm{co}\,H/J}H, Z_P(B))$.*

Proof. Denote for brevity $D := {}^{\mathrm{co}\,H/J}H$. By definition, $\gamma \in \mathrm{Alg}^H(D, P)$. The translation map can be written in terms of γ as follows: $h^{[1]} \otimes h^{[2]} = \gamma(S(h_{(1)})) \otimes_B \gamma(h_{(2)})$. Hence the H-linearity of γ for the Miyashita-Ulbrich action follows directly from the fact that γ is an algebra map. It remains to show that $\gamma(h) \in Z_P(B)$ for all $h \in D$. To this end, note that $D^+ \subseteq J$ and $\Delta(D) \subseteq D \otimes H$ by Lemma 1.6. Now, let $h \in D$ and $b \in B$. Then, using $\nu : D \ni h \mapsto h - \varepsilon(h)1_H \in D^+$, we obtain

$$(1.10) \qquad \gamma(h)b = (h_{(1)} \triangleright b)\gamma(h_{(2)}) = b\gamma(h) + (\nu(h_{(1)}) \triangleright b)\gamma(h_{(2)}) = b\gamma(h).$$

This ends the proof. $\qquad\square$

The next lemma provides a way in which reductions can be combined together in a piecewise trivial comodule algebra.

LEMMA 1.8. *Let H be a Hopf algebra with bijective antipode and J be a Hopf ideal of H such that the antipode of H/J is also bijective. Let P be a piecewise trivial principal H-comodule algebra with a covering $\{\pi_i : P \to P_i\}_{i \in \{1,\dots,N\}}$. Denote $B_i := P_i^{\mathrm{co}\,H}$ and $B := P^{\mathrm{co}\,H}$. Then, if there exists a family of maps $f_i \in \mathrm{Alg}_H^H({}^{coH/J}H, Z_{P_i}(B_i))$, $i \in \{1,\dots,N\}$, such that $\pi_j^i \circ f_i = \pi_i^j \circ f_j$ for all i, j, the following map defined with the help of* (1.3)

$$(1.11) \qquad f : {}^{coH/J}H \longrightarrow P, \quad h \longmapsto \chi^{-1}((f_i(h))_i)$$

is an element of $\mathrm{Alg}_H^H({}^{coH/J}H, Z_P(B))$.

Proof. It is immediate that $f \in \mathrm{Alg}^H({}^{coH/J}H, P)$. Furthermore, for any $h \in {}^{coH/J}H$ and $b \in B$,

$$bf(h) = b\chi^{-1}((f_i(h))_i) = \chi^{-1}((\pi_i(b)f_i(h))_i) = \chi^{-1}((f_i(h)\pi_i(b))_i) = \chi^{-1}((f_i(h))_i)b = f(h)b,$$

so that $f(h) \in Z_P(B)$. Finally, if $\tau : H \to P \otimes_B P$ is the translation map for P, then $(\pi_i \otimes \pi_i) \circ \tau$ is the translation map in P_i and, for any $k \in H$ and $h \in {}^{coH/J}H$, we can compute:

$$\begin{aligned}
(1.12) \qquad k^{[1]}f(h)k^{[2]} &= k^{[1]}\chi^{-1}((f_i(h))_i)k^{[2]} \\
&= \chi^{-1}((\pi_i(k^{[1]})f_i(h)\pi_i(k^{[2]}))_i) \\
&= \chi^{-1}((f_i(Sk_{(1)}hk_{(2)}))_i) \\
&= f(Sk_{(1)}hk_{(2)}).
\end{aligned}$$

Hence f is an element of $\mathrm{Alg}_H^H({}^{coH/J}H, Z_P(B))$. $\qquad\square$

To combine the above two lemmata, we need the following.

LEMMA 1.9. *Let J be a Hopf ideal of H and $\{\gamma_i : H \to P_i\}_{i \in \{1,\dots,N\}}$ be a piecewise trivialisation of a principal H-comodule algebra P. Then, $\forall\, i, j \in \{1, \cdots, N\}$:*

$$(1.13) \qquad T_{ij}(J) = 0 \implies \forall\, h \in {}^{\mathrm{co}\,H/J}H : \pi_j^i(\gamma_i(h)) = \pi_i^j(\gamma_j(h)),$$

where π_j^i's are the canonical surjections of (1.1) and T_{ij}'s are the transition functions of (1.7).

Proof. Denote for brevity $D := {}^{\operatorname{co} H/J}H$. For all i, j, the equality $\pi_j^i(\gamma_i(h)) = \pi_i^j(\gamma_j(h))$ is equivalent to $T_{ij}(h) = \varepsilon(h)$ because $\gamma_j * (\gamma_j \circ S) = \varepsilon = (\gamma_j \circ S) \circ \gamma_j$. On the other hand, $T_{ij}(J) = 0$ by assumption and $D^+ \subseteq J$ by Lemma 1.6, so that, for any $h \in D$, we obtain $T_{ij}(h) = \varepsilon(h) + T_{ij}(h - \varepsilon(h)) = \varepsilon(h)$. $\qquad\square$

The preceding three lemmata combined with Theorem 0.11 yield that $P/Pf(J)$ is an H/J-principal comodule algebra. It remains to show that $P/Pf(J)$ is piecewise trivial. To this end, we apply Lemma 1.3 to show that a covering of P induces a covering of $P/Pf(J)$. For brevity, denote $Pf(J)$ by I. Let $[\cdot] : P \to P/I$ stand for the canonical surjection. Define $\bar{P}_i := P_i/\pi_i(I)$ for all i. The surjections π_i descend to $\bar{\pi}_i : P/I \to \bar{P}_i$. From Lemma 1.3 we conclude that

$$(1.14) \qquad \operatorname{Ker} \bar{\pi}_i = [\operatorname{Ker} \pi_i] = [\operatorname{Ker} \pi_i|_B P] = [\operatorname{Ker} \pi_i|_B][P].$$

On the other hand, since P/I is also a principal comodule algebra and $[B] = [P]^{coH/J}$ by Theorem 0.11, we infer from Lemma 1.3 that $[B] \cap ([\operatorname{Ker} \pi_i|_B][P]) = [\operatorname{Ker} \pi_i|_B]$ for any i. Combining this with (1.14) and remembering $B \cong [B]$ by Theorem 0.11, we compute

$$(1.15)$$
$$\bigcap_{i \in \{1, \cdots, N\}} \operatorname{Ker} \bar{\pi}_i|_{[B]} = \bigcap_{i \in \{1, \cdots, N\}} ([B] \cap \operatorname{Ker} \bar{\pi}_i) = \bigcap_{i \in \{1, \cdots, N\}} [\operatorname{Ker} \pi_i|_B] = [\bigcap_{i \in \{1, \cdots, N\}} \operatorname{Ker} \pi_i|_B] = 0.$$

It also follows that the lattice generated by $\operatorname{Ker} \bar{\pi}_i|_{[B]}$'s is distributive because the lattice generated by $\operatorname{Ker} \pi_i|_B$'s is distributive and $\operatorname{Ker} \bar{\pi}_i|_{[B]} = [\operatorname{Ker} \pi_i|_B] \cong \operatorname{Ker} \pi_i|_B$ for all i. Hence $\{\bar{\pi}_i|_{[B]}\}_i$ is a covering of $[B]$ as needed.

Finally, to prove that the piecewise trivialisation of P induces a piecewise trivialisation of P/I, it suffices to note that the trivialisations (colinear algebra homomorphisms) γ_i descend to trvialisations of $\bar{P}_i$'s. Indeed, since for all i we have $\gamma_i(J) \subseteq \pi_i(I)$, we conclude that there are maps $\bar{\gamma}_i : H/J \ni [h] \mapsto [\gamma_i(h)] \in \bar{P}_i$. They are, clearly, colinear algebra homomorphisms, as needed. Summarising, we have shown that P/I is a piecewise trivial principal H/J-comodule algebra, which ends the proof of one of the implications asserted in Theorem 1.5.

Conversely, now we want to prove that, if we can reduce a principal comodule algebra to a piecewise trivial principal comodule algebra, then the comodule algebra we started from is piecewise trivial in a specific way. Our proof relies on the known fact that the H-prolongation of a reduction of a principal H-comodule algebra is isomorphic with this comodule algebra.

LEMMA 1.10 ([16]). *Let P and Q be principal comodule algebras over Hopf algebras H and K respectively, let $g : H \to K$ be a morphism of Hopf algebras, and let $f : P \to Q$ be an algebra homomorphism that is colinear via g. Assume also that f restricted to P^{coH} gives an isomorphism with Q^{coK}. Then $P \cong Q\square_K H$ as comodule algebras.*

First we consider the cotensor products with trivial comodule algebras.

LEMMA 1.11. *Let $\pi : H \to \bar{H}$ be an epimorphism of Hopf algebras. Assume that $\bar{P}$ is a smash product $\bar{H}$-comodule algebra and $\bar{\gamma} : \bar{H} \to \bar{P}$ is its trivialisation (a colinear algebra homomorphism). Denote $D := {}^{\operatorname{co} \bar{H}}H$ and $B := \bar{P}^{\operatorname{co} \bar{H}}$. Then $\bar{P}\square_{\bar{H}}H$ is a smash product*

H-*comodule algebra and* $\gamma := ((\bar{\gamma} \circ \pi) \otimes \mathrm{id}) \circ \Delta : H \to P \square_{\bar{H}} H$ *is a trivialisation satisfying* $\gamma(k_{(1)}) b \gamma(S(k_{(2)})) = 0$ *for all* $b \in B$ *and* $k \in \mathrm{Ker}\, \pi$.

Proof. For any $b \in B$ and $k \in \mathrm{Ker}\, \pi$, we obtain

$$
(1.16) \qquad \gamma(k_{(1)}) b \gamma(S(k_{(2)})) = \bar{\gamma}(\pi(k_{(1)})) b \bar{\gamma}(\pi(S(k_{(4)}))) \otimes k_{(2)} S(k_{(3)})
$$
$$
= \bar{\gamma}(\pi(k)_{(1)}) b \bar{\gamma}(S(\pi(k)_{(2)})) \otimes 1
$$
$$
= 0.
$$

Also, γ is clearly a colinear algebra homomorphism. $\qquad \square$

Next, we prove a distributivity result for cotensor products that will be useful in the proof of the subsequent lemma.

LEMMA 1.12. *Let* $\bar{P}$ *be a principal* $\bar{H}$-*comodule algebra with* $B := \bar{P}^{\mathrm{co}\,\bar{H}}$, *and let* $\pi : H \to \bar{H}$ *be an epimorphism of Hopf algebras. Assume also that the antipode of* H *is bijective. Let* $\bar{K}_1, \bar{K}_2 \subseteq \bar{P}$ *be ideals and right* $\bar{H}$- *subcomodules in* $\bar{P}$. *Then*

$$
(1.17) \qquad \bar{K}_1 \square_{\bar{H}} H + \bar{K}_2 \square_{\bar{H}} H = (\bar{K}_1 + \bar{K}_2) \square_{\bar{H}} H.
$$

Proof. Let us denote $L_i := B \cap \bar{K}_i$, $i = 1, 2$, for brevity. Using Lemma 1.3, we get

$$
(1.18) \qquad \bar{K}_i = (\bar{K}_i \cap B) \bar{P} = L_i \bar{P}, \quad i = 1, 2.
$$

Similarly, as $\bar{P} \square_{\bar{H}} H$ is a principal H-comodule algebra with $(\bar{P} \square_{\bar{H}} H)^{\mathrm{co}\,H} = B \otimes 1_H$, we can again apply Lemma 1.3 to obtain

$$
(1.19)
$$
$$
\bar{K}_i \square_{\bar{H}} H = \left((B \otimes 1_H) \cap \bar{K}_i \square_{\bar{H}} H \right) (\bar{P} \square_{\bar{H}} H) = ((B \cap \bar{K}_i) \bar{P}) \square_{\bar{H}} H = L_i \bar{P} \square_{\bar{H}} H, \quad i = 1, 2.
$$

Hence

$$
\bar{K}_1 \square_{\bar{H}} H + \bar{K}_2 \square_{\bar{H}} H = L_1 \bar{P} \square_{\bar{H}} H + L_2 \bar{P} \square_{\bar{H}} H
$$
$$
= (L_1 + L_2) \bar{P} \square_{\bar{H}} H
$$
$$
= ((L_1 + L_2) \bar{P}) \square_{\bar{H}} H
$$
$$
= (L_1 \bar{P} + L_2 \bar{P}) \square_{\bar{H}} H
$$
$$
(1.20) \qquad = (K_1 + K_2) \square_{\bar{H}} H,
$$

as needed. $\qquad \square$

Now we are ready to generalize the Lemma 1.11 from trivial comodule algebras to piecewise trivial comodule algebras.

LEMMA 1.13. *Let* $\bar{P}$ *be a piecewise trivial principal* $\bar{H}$-*comodule algebra with* $B := \bar{P}^{\mathrm{co}\,\bar{H}}$, *let* $\{\bar{\pi}_i : \bar{P} \to \bar{P}_i\}_{i \in \{1, \cdots, N\}}$ *be a covering of* $\bar{P}$, *and let* $\{\bar{\gamma}_i : \bar{H} \to \bar{P}_i\}_{i \in \{1, \cdots, N\}}$ *be a family of trivialisations (colinear algebra homomorphisms). Assume also that* $\pi : H \to \bar{H}$ *is an epimorphism of Hopf algebras, the antipode of* H *is bijective, and* H *is a principal left* $\bar{H}$-*comodule algebra. Then* $\bar{P} \square_{\bar{H}} H$ *is a piecewise trivial principal comodule algebra for the covering*

$$
(1.21) \qquad \{\bar{\pi}_i \square_{\bar{H}} \mathrm{id}_H : \bar{P} \square_{\bar{H}} H \longrightarrow \bar{P}_i \square_{\bar{H}} H\}_{i \in \{1, \cdots, N\}},
$$

the maps

$$(1.22) \qquad \{H \ni k \xmapsto{\gamma_i} \bar{\gamma}_i(\pi(k_{(1)})) \otimes k_{(2)} \in \bar{P}_i \square_{\bar{H}} H\}_{i \in \{1, \cdots, N\}}$$

are trivialisations satisfying $\gamma_i(k_{(1)})b\gamma_i(S(k_{(2)})) = 0$ *for all* $b \in \bar{P}_i^{\operatorname{co}\bar{H}} \otimes 1$, $k \in \operatorname{Ker}\pi$, *and the associated transition functions* T_{ij} *(see* (1.7)*) fulfill* $T_{ij}(\operatorname{Ker}\pi) = 0$ *for all* $i, j \in \{1, \cdots, N\}$.

Proof. First note that since the principality of H implies the coflatness of H as a left $\bar{H}$-comodule [16, Theorem II.3.26], it follows that the maps $\bar{\pi}_i \otimes \operatorname{id}$ are all surjective. Because $\{\bar{\pi}_i|_B\}_i$ is a covering of B it is immediate that $\{\bar{\pi}_i \otimes \operatorname{id}_H|_{B \otimes 1_H}\}_i$ is a covering of $B \otimes 1_H = (\bar{P}\square_{\bar{H}}H)^{\operatorname{co}H}$.

Next, from Lemma 1.11 we conclude that all the trivialisations (1.22) satisfy $\gamma_i(k_{(1)})b\gamma_i(S(k_{(2)})) = 0$ for all $b \in \bar{P}_i^{\operatorname{co}\bar{H}} \otimes 1$, $k \in \operatorname{Ker}\pi$. Finally, we prove the desired property of the associated transition functions. The left exactness of the cotensor functor implies that $\operatorname{Ker}(\bar{\pi}_i\square_{\bar{H}}\operatorname{id}_H) = (\operatorname{Ker}\bar{\pi}_i)\square_{\bar{H}}H$. Combining this with Lemma 1.12 and the left coflatness of H over $\bar{H}$, we obtain the canonical isomorphism φ

$$(\bar{P}\square_{\bar{H}}H)/(\operatorname{Ker}(\bar{\pi}_i\square_{\bar{H}}\operatorname{id}_H) + \operatorname{Ker}(\bar{\pi}_j\square_{\bar{H}}\operatorname{id}_H)) = (\bar{P}\square_{\bar{H}}H)/(\operatorname{Ker}(\bar{\pi}_i)\square_{\bar{H}}H + \operatorname{Ker}(\bar{\pi}_j)\square_{\bar{H}}H)$$
$$= (\bar{P}\square_{\bar{H}}H)/(\operatorname{Ker}\bar{\pi}_i + \operatorname{Ker}\bar{\pi}_j)\square_{\bar{H}}H$$
$$(1.23) \qquad\qquad \cong (\bar{P}/(\operatorname{Ker}\bar{\pi}_i + \operatorname{Ker}\bar{\pi}_j))\square_{\bar{H}}H.$$

Hence we conclude that $\pi_j^i = \varphi^{-1} \circ (\bar{\pi}_j^i \otimes \operatorname{id}_H)$ for all i and j. Therefore, we can write the transition functions (see (1.7)) as

$$T_{ij}(k) = \pi_j^i(\gamma_i(k_{(1)}))\pi_i^j(\gamma_j(S(k_{(2)})))$$
$$= \varphi^{-1}(\bar{\pi}_j^i(\bar{\gamma}_i(\pi(k_{(1)})))\bar{\pi}_i^j(\bar{\gamma}_j(\pi(S(k_{(4)})))) \otimes k_{(2)}S(k_{(3)}))$$
$$(1.24) \qquad\qquad = \varphi^{-1}(\bar{\pi}_j^i(\bar{\gamma}_i(\pi(k_{(1)})))\bar{\pi}_i^j(\bar{\gamma}_j(\pi(S(k_{(2)})))) \otimes 1_K).$$

Now the equality $T_{ij}(J) = 0$ for any i and j follows from the fact that $J := \operatorname{Ker}\pi$ is a Hopf ideal. $\qquad\square$

Summarising, it follows from Lemma 1.13 and Lemma 1.10 that, if a principal comodule algebra P is reducible to a piecewise trivial principal comodule algebra $\bar{P}$, then there exists a trivialisation of P satisfying the two conditions of the theorem.

2. Noncommutative bundles over the Toeplitz deformation of $\mathbb{R}P^2$

2.1. **A quantum real projective space.** In [14] a new type of a noncommutative deformation of complex projective spaces was constructed. The construction is based on the idea of covering a complex projective space by Cartesian powers of closed discs (a compact restriction of the canonical affine covering). Then discs are replaced by quantum discs [KL93] given in terms of the Toeplitz algebra $\mathcal{T}$.

For real projective spaces $\mathbb{R}P^N$, $N - 1 \in \mathbb{N}$, a suitable compact restriction of the canonical affine covering is given by cubes I^N, where I is the real unit disc, i.e. $I := [-1, 1]$. Now we

replace I^{2k} by $\mathcal{T}^{\otimes k}$ and I^{2k+1} by $\mathcal{T}^{\otimes k} \otimes_{\min} C(I)$. Thus in the real case we are forced to consider the even and odd dimension separately.

Here we carry out the aforementioned construction for $N = 2$. Hence the C^*-algebra of our quantum $\mathbb{R}P^2$ will be a triple-pullback C^*-algebra obtained from three Toeplitz algebras viewed this time as the C^*-algebras of quantum squares rather than quantum discs. We consider the Toeplitz algebra $\mathcal{T}$ as the universal C^*-algebra generated by an isometry s, and the symbol map given by the assignment $\sigma \colon \mathcal{T} \ni s \mapsto \widetilde{u} \in C(S^1)$, where $\widetilde{u}$ is the unitary function generating $C(S^1)$. Now we are ready "to square the bounadry circle" of the quantum disc with the help of the following two maps

$$(2.1) \quad \mathbb{Z}/2\mathbb{Z} \times I \ni (k, t) \xmapsto{\delta_1} e^{i\pi(\frac{1}{4}kt + \frac{1}{2}k + \frac{3}{2})} \in S^1, \quad I \times \mathbb{Z}/2\mathbb{Z} \ni (t, k) \xmapsto{\delta_2} e^{i\pi(-\frac{1}{4}kt - \frac{1}{2}k + 1)} \in S^1,$$

and their pullbacks

$$(2.2) \qquad \delta_1^* \colon C(S^1) \longrightarrow C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I), \quad \delta_2^* \colon C(S^1) \longrightarrow C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}).$$

We will denote for brevity $\sigma_i := \delta_i^* \circ \sigma$, $i = 1, 2$. Each of the maps δ_i can be understood as a parametrisation of two appropriate quarters of S^1 as shown on the pictures below:
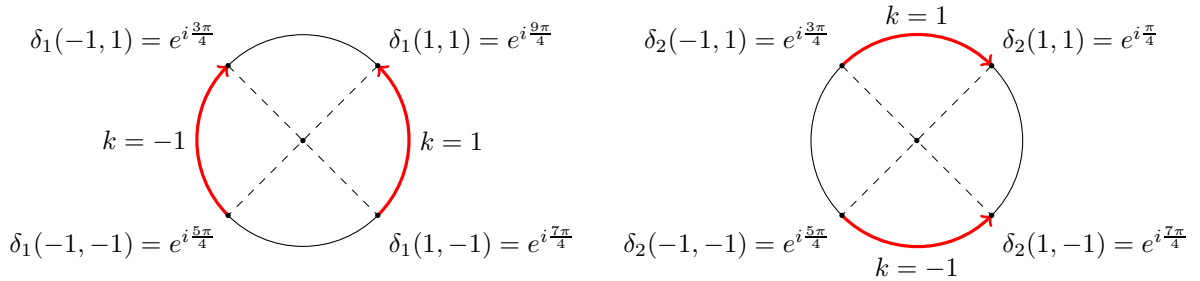
We view S^1 and I as $\mathbb{Z}/2\mathbb{Z}$-spaces via multiplication by ± 1. Then $\mathbb{Z}/2\mathbb{Z} \times I$ and $I \times \mathbb{Z}/2\mathbb{Z}$ are $\mathbb{Z}/2\mathbb{Z}$-spaces with the diagonal action. Accordingly, $C(I)$, $C(S^1)$, $C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I)$ and $C(I) \otimes C(\mathbb{Z}/2\mathbb{Z})$ are right $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebras with coactions given by the pullbacks of respective $\mathbb{Z}/2\mathbb{Z}$-actions. Denote by u the generator $C(\mathbb{Z}/2\mathbb{Z})$ given by $u(\pm 1) := \pm 1$. Then the assignment $s \mapsto s \otimes u$ makes $\mathcal{T}$ a $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra. (This coaction corresponds to the $\mathbb{Z}/2\mathbb{Z}$-action given by $\alpha_{-1}^{\mathcal{T}}(s) = -s$.) It is easy to verify that the maps δ_i, $i = 1, 2$, are $\mathbb{Z}/2\mathbb{Z}$-equivariant, so that their pullbacks δ_i^*'s are right $C(\mathbb{Z}/2\mathbb{Z})$-comodule maps. Also, since the symbol map σ is a right $C(\mathbb{Z}/2\mathbb{Z})$-comodule map, so are σ_i's.

Now we are ready to define the C^*-algebra $C(\mathbb{R}P_{\mathcal{T}}^2)$ of our Toeplitz deformation of the real projective plane. We take three copies of the Toeplitz algebra $\mathcal{T}$, distinguish them by subscripts

for clarity, and write the building blocks of a triple pullback diagram as follows:

$$
\begin{array}{cc}
\mathcal{T}_0 & \mathcal{T}_1 \\
\sigma_1 \downarrow & \downarrow \sigma_1 \\
C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I) \xleftarrow{\Psi_{01}} C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I), &
\end{array}
$$

(2.3)

$$
\begin{array}{cc}
\mathcal{T}_0 & \mathcal{T}_2 \\
\sigma_2 \downarrow & \downarrow \sigma_1 \\
C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \xleftarrow{\Psi_{02}} C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I),
\end{array}
$$

$$
\begin{array}{cc}
\mathcal{T}_1 & \mathcal{T}_2 \\
\sigma_2 \downarrow & \downarrow \sigma_2 \\
C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \xleftarrow{\Psi_{12}} C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}).
\end{array}
$$

Here the isomorphisms Ψ_{ij} are given by formulae analogous to the formulae used in [14], that is:

(2.4)
$$
\begin{aligned}
C(\mathbb{Z}_2) \otimes C(I) \ni u \otimes x &\xrightarrow{\Psi_{01}} x_{(1)}u \otimes x_{(0)} \in C(\mathbb{Z}_2) \otimes C(I), \\
C(\mathbb{Z}_2) \otimes C(I) \ni u \otimes x &\xrightarrow{\Psi_{02}} x_{(0)} \otimes x_{(1)}u \in C(I) \otimes C(\mathbb{Z}_2), \\
C(I) \otimes C(\mathbb{Z}_2) \ni x \otimes u &\xrightarrow{\Psi_{12}} x_{(0)} \otimes x_{(1)}u \in C(I) \otimes C(\mathbb{Z}_2).
\end{aligned}
$$

Putting all this together, we define the C^*-algebra

(2.5)
$$
\begin{aligned}
C(\mathbb{R}P_{\mathcal{T}}^2) := \big\{ (t_0, t_1, t_2) \in \mathcal{T}^3 \mid\ & \sigma_1(t_0) = (\Psi_{01} \circ \sigma_1)(t_1), \\
& \sigma_2(t_0) = (\Psi_{02} \circ \sigma_1)(t_2), \\
& \sigma_2(t_1) = (\Psi_{12} \circ \sigma_2)(t_2) \big\}.
\end{aligned}
$$

2.2. **From $\mathbb{R}P_{\mathcal{T}}^2$ to quantum 2-sphere.** The usual way of constructing real projective spaces is by taking $\mathbb{Z}/2\mathbb{Z}$-quotients of spheres. Here we reverse this procedure, i.e., we treat projective spaces as primary objects, and construct spheres from them. More precisely, since each cube covering a real projective space is contractible, any principal bundle over such a cube must be trivial. Consequently, as the fiber of each principal bundle $S^N \to \mathbb{R}P^N$, $N - 1 \in \mathbb{N}$, is $\mathbb{Z}/2\mathbb{Z}$, we can assemble any sphere by appropriate glueing of pairs of cubes. In particular, for $N = 2$, we construct the topological 2-sphere by assembling three pairs of squares to the boundary of a cube.

Our aim is to construct a quantum sphere $S_{\mathbb{R}\mathcal{T}}^2$ as a $\mathbb{Z}/2\mathbb{Z}$-bundle over $\mathbb{R}P_{\mathcal{T}}^2$. To this end, we take $\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z})$ as basic ingredients, and write building blocks of a triple-pullback diagram

as follows:

$$\begin{array}{ccc}
\mathcal{T}_0 \otimes C(\mathbb{Z}/2\mathbb{Z}) & & \mathcal{T}_1 \otimes C(\mathbb{Z}/2\mathbb{Z}) \\
{\scriptstyle \sigma_1 \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \sigma_1 \otimes \mathrm{id}} \\
C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\widetilde{\Phi}_{01}} & C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I) \otimes C(\mathbb{Z}/2\mathbb{Z})\,,
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}_0 \otimes C(\mathbb{Z}/2\mathbb{Z}) & & \mathcal{T}_2 \otimes C(\mathbb{Z}/2\mathbb{Z}) \\
{\scriptstyle \sigma_2 \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \sigma_1 \otimes \mathrm{id}} \\
C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \otimes C(\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\widetilde{\Phi}_{02}} & C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I) \otimes C(\mathbb{Z}/2\mathbb{Z})\,,
\end{array}$$

(2.6)
$$\begin{array}{ccc}
\mathcal{T}_1 \otimes C(\mathbb{Z}/2\mathbb{Z}) & & \mathcal{T}_2 \otimes C(\mathbb{Z}/2\mathbb{Z}) \\
{\scriptstyle \sigma_2 \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \sigma_2 \otimes \mathrm{id}} \\
C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \otimes C(\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\widetilde{\Phi}_{12}} & C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \otimes C(\mathbb{Z}/2\mathbb{Z})\,.
\end{array}$$

Here the isomorphisms $\widetilde{\Phi}_{ij}$ are given by the formulae (see [9, eq. 9])

(2.7) $$\widetilde{\Phi}_{ij}(a \otimes b \otimes h) := \Psi_{ij}(a \otimes b) T_{ij}(h_{(1)}) \otimes h_{(2)}\,, \quad i,j \in \{0,1,2\}, \quad i < j,$$

for some transitions functions T_{ij} that will be determined later. Now we can define our triple-pullback C^*-algebra in the following way

$$\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})} := \big\{(t_i \otimes u_i)_i \in (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))^3 \mid (\sigma_1 \otimes \mathrm{id})(t_0 \otimes u_0) = \big(\widetilde{\Phi}_{01} \circ (\sigma_1 \otimes \mathrm{id})\big)(t_1 \otimes u_1),$$
$$(\sigma_2 \otimes \mathrm{id})(t_0 \otimes u_0) = \big(\widetilde{\Phi}_{02} \circ (\sigma_1 \otimes \mathrm{id})\big)(t_2 \otimes u_2),$$
(2.8)
$$(\sigma_2 \otimes \mathrm{id})(t_1 \otimes u_1) = \big(\widetilde{\Phi}_{12} \circ (\sigma_2 \otimes \mathrm{id})\big)(t_2 \otimes u_2)\big\}.$$

If we consider the natural $\mathbb{Z}/2\mathbb{Z}$-actions on the rightmost tensorands of the components of $\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}$, all maps in the diagram (2.6) are $\mathbb{Z}/2\mathbb{Z}$-equivariant C^*-homomorphisms. Thus we obtain a $\mathbb{Z}/2\mathbb{Z}$-action on $\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}$ such that its fixed-point subalgebra satisfies

(2.9) $$\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}^{\mathbb{Z}/2\mathbb{Z}} = C(\mathbb{R}P^2_{\mathcal{T}}) \otimes \mathbb{C}.$$

Trading the above action for coaction, we can view the components of $C(S^2_{\mathbb{R}\mathcal{T}})$ as trivial $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebras. However, to see that the quantum real projective space $\mathbb{R}P^2_{\mathcal{T}}$ corresponds to the quotient of $S^2_{\mathbb{R}\mathcal{T}}$ by the antipodal $\mathbb{Z}/2\mathbb{Z}$-action, we need to transform $\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}$ into an appropraite isomorphic $\mathbb{Z}/2\mathbb{Z}$-C^*-algebra. To this end, we need to gauge the aforementioned $\mathbb{Z}/2\mathbb{Z}$-actions on components to the diagonal $\mathbb{Z}/2\mathbb{Z}$-actions thereon. We transform the former into the latter by conjugating all maps of (2.6) by the gauge transformation of the form

(2.10) $$g_B : B \otimes C(\mathbb{Z}/2\mathbb{Z}) \ni b \otimes h \longmapsto b_{(0)} \otimes b_{(1)}h \in B \otimes C(\mathbb{Z}/2\mathbb{Z}).$$

To define the diagonal action on the right-hand side, we view B as one of the following $\mathbb{Z}/2\mathbb{Z}$-C^*-algebras: $C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I)$ with the diagonal antipodal action, $C(I) \otimes C(\mathbb{Z}/2\mathbb{Z})$ again with the diagonal antipodal action, or $\mathcal{T}$ with the $\mathbb{Z}/2\mathbb{Z}$-action given by $\alpha^{\mathcal{T}}_{-1}(s) = -s$.

For brevity, in what follows we will omit the subscript distinguishing various maps g whenever it is implied by the context. To compute the result of conjugation of morphisms in (2.6) with maps g, first we note that $g = g^{-1}$ because the antipode of $C(\mathbb{Z}/2\mathbb{Z})$ is equal to the identity function. Next, it is immediate to verify that, due to the right $C(\mathbb{Z}/2\mathbb{Z})$-colinearity σ_i's, we obtain

$$(2.11) \qquad g \circ (\sigma_1 \otimes \mathrm{id}) \circ g = (\sigma_1 \otimes \mathrm{id}) \quad \text{and} \quad g \circ (\sigma_2 \otimes \mathrm{id}) \circ g = (\sigma_2 \otimes \mathrm{id}).$$

Furthermore, let us define $\Phi_{ij} := g \circ \tilde{\Phi}_{ij} \circ g$, and compute:

$$
\begin{aligned}
\Phi_{01}(h \otimes p \otimes k) &= (g \circ \tilde{\Phi}_{01} \circ g)(h \otimes p \otimes k) \\
&= (g \circ \tilde{\Phi}_{01})(h_{(1)} \otimes p_{(0)} \otimes h_{(2)}p_{(1)}k) \\
&= g\big(\Psi_{01}(h_{(1)} \otimes p_{(0)})T_{01}(h_{(2)}p_{(1)}k_{(1)}) \otimes h_{(3)}p_{(2)}k_{(2)}\big) \\
&= g\big((h_{(1)}p_{(1)} \otimes p_{(0)})T_{01}(h_{(2)}p_{(2)}k_{(1)}) \otimes h_{(3)}p_{(3)}k_{(2)}\big) \\
&= (h_{(1)}p_{(2)} \otimes p_{(0)})T_{01}(h_{(3)}p_{(4)}k_{(1)})_{(0)} \otimes h_{(2)}p_{(3)}p_{(1)}T_{01}(h_{(3)}p_{(4)}k_{(1)})_{(1)}h_{(4)}p_{(5)}k_{(2)} \\
&= (h_{(1)}p_{(1)} \otimes p_{(0)})T_{01}(h_{(2)}p_{(5)}k_{(1)})_{(0)} \otimes h_{(3)}h_{(4)}p_{(2)}p_{(3)}T_{01}(h_{(2)}p_{(5)}k_{(1)})_{(1)}p_{(4)}k_{(2)} \\
(2.12) \qquad &= (h_{(1)}p_{(1)} \otimes p_{(0)})T_{01}(h_{(2)}p_{(3)}k_{(1)})_{(0)} \otimes T_{01}(h_{(2)}p_{(3)}k_{(1)})_{(1)}p_{(2)}k_{(2)}.
\end{aligned}
$$

The penultimate line above follows from the commutativity and cocommutativity of $C(\mathbb{Z}/2\mathbb{Z})$. The last equality is a consequence of $h_{(1)}h_{(2)} = \varepsilon(h)$ for all $h \in C(\mathbb{Z}/2\mathbb{Z})$. The computations for Φ_{02} and Φ_{12} are similar.

Finally, we determine the transition functions T_{ij}, so that $C(S^2_{\mathbb{R}\mathcal{T}})$ is indeed a noncommutative deformation of $C(S^2)$. To this end, we observe that Φ_{01} is the pullback of the following map:

$$(2.13) \qquad \mathbb{Z}/2\mathbb{Z} \times I \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Psi^*_{01}} \mathbb{Z}/2\mathbb{Z} \times I \times \mathbb{Z}/2\mathbb{Z},$$

$$(a, t, c) \longmapsto (af_{01}(ac, tc),\ atcf_{01}(ac, ct),\ cf_{01}(ac, tc)).$$

Here $f_{01} : \mathbb{Z}/2\mathbb{Z} \times I \to \mathbb{Z}/2\mathbb{Z}$ is the map whose pullback is T_{01}. Much in the same way, we note that Φ_{02} is the pullback of

$$(2.14) \qquad \mathbb{Z}/2\mathbb{Z} \times I \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Psi^*_{02}} I \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$(a, t, c) \longmapsto (atcf_{02}(ac, tc),\ af_{02}(ac, tc),\ cf_{02}(ac, tc)).$$

and Φ_{12} is the pullback of

$$(2.15) \qquad I \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\Psi^*_{12}} I \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

$$(t, a, c) \longmapsto (atcf_{12}(ac, tc),\ af_{12}(ac, tc),\ cf_{12}(ac, tc)).$$

The continuity of f_{ij}'s implies that they are independent of their continuous argument. Hence we have to choose only between four possible functions: id, $-\mathrm{id}$, 1, -1. We choose $f_{ij} = \mathrm{id}$ for all $i, j \in \{0, 1, 2\}$, $i < j$. Thus we obtain

$$
\begin{aligned}
\Phi_{01}(h \otimes p \otimes k) &= k \otimes p \otimes h, \\
\Phi_{02}(h \otimes p \otimes k) &= p \otimes k \otimes h, \\
(2.16) \qquad \Phi_{12}(p \otimes h \otimes k) &= p \otimes k \otimes h.
\end{aligned}
$$

We are now ready to define the following triple-pullback C^*-algebra:

$$C(S^2_{\mathbb{R}\mathcal{T}}) := \big\{(t_i \otimes u_i)_i \in (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))^3 \mid (\sigma_1 \otimes \mathrm{id})(t_0 \otimes u_0) = \big(\Phi_{01} \circ (\sigma_1 \otimes \mathrm{id})\big)(t_1 \otimes u_1),$$
$$(\sigma_2 \otimes \mathrm{id})(t_0 \otimes u_0) = \big(\Phi_{02} \circ (\sigma_1 \otimes \mathrm{id})\big)(t_2 \otimes u_2),$$
$$(2.17) \qquad\qquad (\sigma_2 \otimes \mathrm{id})(t_1 \otimes u_1) = \big(\Phi_{12} \circ (\sigma_2 \otimes \mathrm{id})\big)(t_2 \otimes u_2)\big\}.$$

Since the diagonal and the rightmost $\mathbb{Z}/2\mathbb{Z}$-actions on the components of $C(S^2_{\mathbb{R}\mathcal{T}})$ and $\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}$ respectively are intertwined by C^*-isomorphisms, we conclude that they are isomorphic as $\mathbb{Z}/2\mathbb{Z}$-C^*-algebras. Consequently, their invariant subalgebras are naturally isomorphic. Combining this with (2.9), we obtain an isomorphism of C^*-algebras.

$$(2.18) \qquad\qquad C(S^2_{\mathbb{R}\mathcal{T}})^{\mathbb{Z}/2\mathbb{Z}} \cong C(\mathbb{R}P^2_{\mathcal{T}}).$$

One can check that replacing in the foregoing construction of $C(S^2_{\mathbb{R}\mathcal{T}})$ the Toeplitz algebra $\mathcal{T}$ by the algebra $C(D)$ of continuous functions on the unit disc, yields a C^*-algebra isomorphic with $C(S^2)$. Also, the $\mathbb{Z}/2\mathbb{Z}$-action on $C(S^2_{\mathbb{R}\mathcal{T}})$, which is given by the diagonal action on each component, becomes precisely the the pullback of the diagonal action on S^2. The isomorphism is given by rounding the boundary of a cube to the unit sphere. Indeed, using the notation

$$(2.19) \quad \mathcal{T}_{i,j} := (\mathrm{id} \otimes \mathrm{ev}_j)(\mathcal{T}_i \otimes C(\mathbb{Z}/2\mathbb{Z})), \quad E_{1,i} := (\mathrm{ev}_i \otimes \mathrm{id}) \circ \sigma_1, \quad E_{2,i} := (\mathrm{id} \otimes \mathrm{ev}_i) \circ \sigma_2,$$

allows us to verify it with the help of the following picture
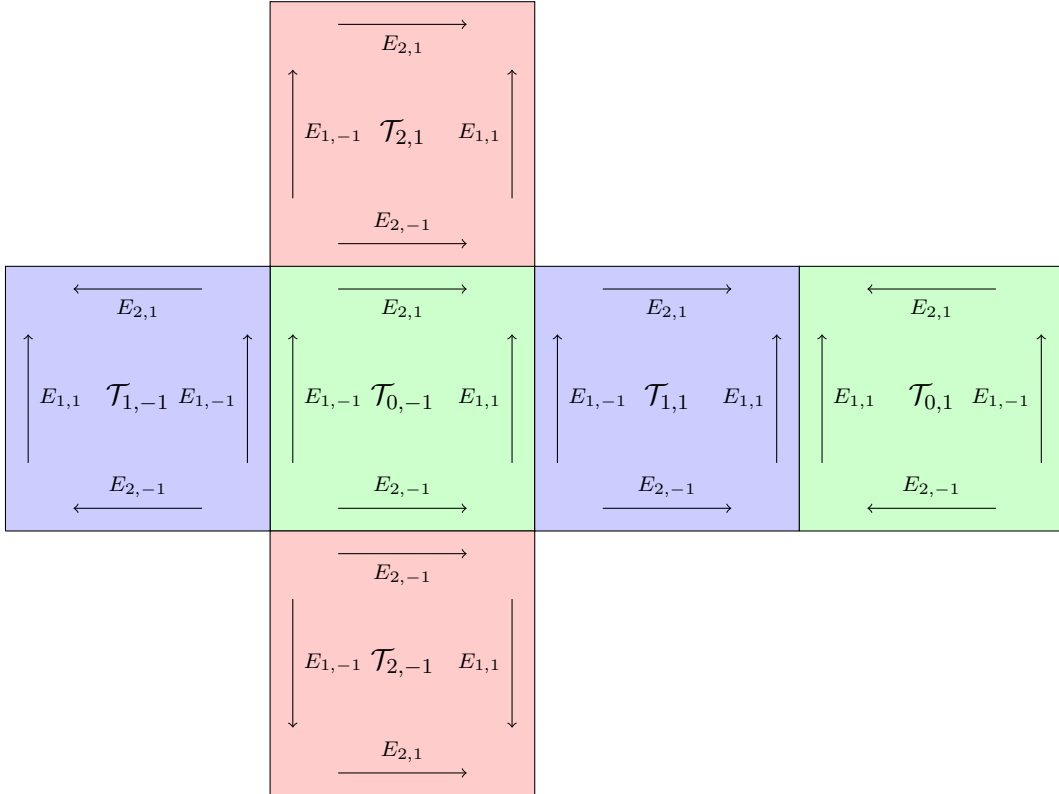
REMARK 2.1. It is worth mentioning that the choice $f_{ij} = 1$ for all $i, j \in \{0, 1, 2\}$, $i < j$ would yield the C^*-algebra $C(\mathbb{R}P^2_{\mathcal{T}}) \otimes C(\mathbb{Z}/2\mathbb{Z})$.

2.3. **The quantum $\mathbb{Z}/2\mathbb{Z}$-principal bundle $S^2_{\mathbb{R}\mathcal{T}} \to \mathbb{R}P^2_{\mathcal{T}}$.** To prove that the $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra $C(S^2_{\mathbb{R}\mathcal{T}})$ constructed above as a triple-pullback comodule algebra is principal, first we need to show that all restrictions $C(S^2_{\mathbb{R}\mathcal{T}}) \to (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))_i$ of the canonical surjections remain surjective. A sufficient condition for the aforementioned surjectivity is given in the following technical proposition.

PROPOSITION 2.2. [8, Prop. 9] *Let us denote for brevity $\underline{N} := \{0, \dots, N\}$. Let $\{B_i\}_{i \in \underline{N}}$ and $\{B_{ij}\}_{i,j \in \underline{N}, i \neq j}$ be two families of algebras such that $B_{ij} = B_{ji}$, and let $\{\pi^i_j : B_i \to B_{ij}\}_{ij}$ be a family of surjective algebra maps whose kernels generate a distributive lattice of ideals. Also, let $\pi_i : B \to B_i$, $i \in \underline{N}$, be the restrictions to*

$$B := \{(b_i)_i \in \textstyle\prod_{i \in \underline{N}} B_i \mid \pi^i_j(b_i) = \pi^j_i(b_j), \ \forall i, j \in \underline{N}, \ i \neq j\}$$

of the canonical projections. Assume that, for all triples of distinct indices $i, j, k \in \underline{N}$, the following conditions hold:

(1) $\pi^i_j(\operatorname{Ker} \pi^i_k) = \pi^j_i(\operatorname{Ker} \pi^j_k)$;

(2) *the isomorphisms $\pi^{ij}_k : B_i/(\operatorname{Ker} \pi^i_j + \operatorname{Ker} \pi^i_k) \longrightarrow B_{ij}/\pi^i_j(\operatorname{Ker} \pi^i_k)$ defined as*

$$b_i + \operatorname{Ker} \pi^i_j + \operatorname{Ker} \pi^i_k \longmapsto \pi^i_j(b_i) + \pi^i_j(\operatorname{Ker} \pi^i_k)$$

$$\text{satisfy} \quad (\pi^{ik}_j)^{-1} \circ \pi^{ki}_j = (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k \circ (\pi^{jk}_i)^{-1} \circ \pi^{kj}_i.$$

Then, $\forall \ (b_i)_{i \in I} \in \prod_{i \in I} B_i$, $I \subseteq \underline{N}$, such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, $\forall \ i, j \in I$, $i \neq j$,

$$\exists \ (c_i)_{i \in \underline{N}} \in \textstyle\prod_{i \in \underline{N}} B_i : \ \pi^i_j(c_i) = \pi^j_i(c_j), \ \forall \ i, j \in \underline{N}, \ i \neq j, \ \text{and} \ c_i = b_i, \forall \ i \in I.$$

Our task now is to check that our multipullback construction of $C(S^2_{\mathbb{R}\mathcal{T}})$ satisfies the assumptions of Proposition 2.2. The distributivity condition is automatically satisfied because here we work with C^*-ideals, and the lattices of C^*-ideals are always distributive. We begin by defining certain auxiliary maps $\hat{\varphi}_1, \hat{\varphi}_2 \in C(S^1)$ by the formulae:

(2.20) $\quad \hat{\varphi}_1(e^{i\theta}) := \begin{cases} 2 - \frac{4}{\pi}\theta & \text{if} \quad \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\[4pt] -1 & \text{if} \quad \theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \\[4pt] \frac{4}{\pi}\theta - 6 & \text{if} \quad \theta \in [\frac{5\pi}{4}, \frac{7\pi}{4}] \\[4pt] 1 & \text{if} \quad \theta \in [\frac{7\pi}{4}, \frac{9\pi}{4}] \end{cases}, \quad \hat{\varphi}_2(e^{i\theta}) := \begin{cases} 1 & \text{if} \quad \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \\[4pt] 4 - \frac{4}{\pi}\theta & \text{if} \quad \theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}] \\[4pt] -1 & \text{if} \quad \theta \in [\frac{5\pi}{4}, \frac{7\pi}{4}] \\[4pt] \frac{4}{\pi}\theta - 8 & \text{if} \quad \theta \in [\frac{7\pi}{4}, \frac{9\pi}{4}] \end{cases}.$

One immediately sees that

(2.21) $\qquad \hat{\varphi}_1, \hat{\varphi}_2 : S^1 \longrightarrow [-1, 1], \quad \hat{\varphi}_1(-z) = -\hat{\varphi}_1(z), \quad \hat{\varphi}_2(-z) = -\hat{\varphi}_2(z).$

Next, let us denote by $\iota_I \in C(I)$ the inclusion given by $\iota_I(t) := t$, where $t \in I := [-1, 1]$. Recalling that u is the generator of $C(\mathbb{Z}/2\mathbb{Z})$ given by $u(\pm 1) = \pm 1$, and remembering (2.1), one easily verifies the following properties of $\hat{\varphi}_i$'s:

(2.22) $\hat{\varphi}_1 \circ \delta_1 = u \otimes 1_{C(I)}, \quad \hat{\varphi}_2 \circ \delta_2 = 1_{C(I)} \otimes u, \quad \hat{\varphi}_1 \circ \delta_2 = \iota_I \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})}, \quad \hat{\varphi}_2 \circ \delta_1 = 1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes \iota_I.$

Furthermore, we need to define unital and right $C(\mathbb{Z}/2\mathbb{Z})$-colinear splittings

(2.23) $\qquad \hat{\omega}_1 : C(\mathbb{Z}/2\mathbb{Z}) \otimes C(I) \longrightarrow C(S^1), \qquad \hat{\omega}_2 : C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \longrightarrow C(S^1),$

of δ_1^* and δ_2^* respectively. To this end, take any $h \in C(I)$ and set

$$\hat{\omega}_1(1 \otimes h) := h \circ \hat{\varphi}_2, \quad \hat{\omega}_1(u \otimes h) := \hat{\varphi}_1 \cdot (h \circ \hat{\varphi}_2),$$

(2.24)
$$\hat{\omega}_2(h \otimes 1) := h \circ \hat{\varphi}_1, \quad \hat{\omega}_2(h \otimes u) := \hat{\varphi}_2 \cdot (h \circ \hat{\varphi}_1).$$

Here $\cdot$ stands for the pointwise multiplication in $C(S^1)$. The right colinearity of $\hat{\omega}_i$'s follows immediately from (2.21), and it is straightforward to check that $\hat{\omega}_i$'s are splittings, i.e., $\delta_1^* \circ \hat{\omega}_1 = \text{id}$, $\delta_2^* \circ \hat{\omega}_2 = \text{id}$. Indeed, take any $h \in C(I)$ and use (2.22) to compute:

$$(\delta_1^* \circ \hat{\omega}_1)(1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes h) = h \circ \hat{\varphi}_2 \circ \delta_1 = h \circ (1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes \iota_I) = (1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes h),$$

(2.25)
$$(\delta_1^* \circ \hat{\omega}_1)(u \otimes h) = (\hat{\varphi}_1 \circ \delta_1) \cdot (h \circ \hat{\varphi}_2 \circ \delta_1) = (u \otimes 1_{C(I)}) \cdot (1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes h) = u \otimes h.$$

The case of $\delta_2^* \circ \hat{\omega}_2$ is analogous.

To prove certain additional properties of ω_1 and ω_2, let us denote by $\imath_{\mathbb{Z}/2\mathbb{Z}} : \mathbb{Z}/2\mathbb{Z} \to I$ the inclusion map, by $\imath_{\mathbb{Z}/2\mathbb{Z}}^* : C(I) \to C(\mathbb{Z}/2\mathbb{Z})$ its pullback, and by $\imath_*^{\mathbb{Z}/2\mathbb{Z}} : C(\mathbb{Z}/2\mathbb{Z}) \to C(I)$ the right $C(\mathbb{Z}/2\mathbb{Z})$-colinear splitting of $\imath_{\mathbb{Z}/2\mathbb{Z}}^*$ defined by the formula

(2.26)
$$\imath_*^{\mathbb{Z}/2\mathbb{Z}}(1_{C(\mathbb{Z}/2\mathbb{Z})}) = 1_{C(I)}, \quad \imath_*^{\mathbb{Z}/2\mathbb{Z}}(u) = \imath_I.$$

Now we are ready to verify that

(2.27)
$$\delta_2^* \circ \hat{\omega}_1 = \imath_*^{\mathbb{Z}/2\mathbb{Z}} \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*, \quad \delta_1^* \circ \hat{\omega}_2 = \imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}.$$

To this end, again take any $h \in C(I)$ and use (2.22) to compute

$$(\delta_2^* \circ \hat{\omega}_1)(1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes h) = h \circ \hat{\varphi}_2 \circ \delta_2 = h \circ (1_{C(I)} \otimes u)$$
$$= \imath_*^{\mathbb{Z}/2\mathbb{Z}}(1_{C(\mathbb{Z}/2\mathbb{Z})}) \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(h),$$
$$(\delta_2^* \circ \hat{\omega}_1)(u \otimes h) = (\hat{\varphi}_1 \circ \delta_2) \cdot (h \circ \hat{\varphi}_2 \circ \delta_2) = (\imath_I \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})}) \cdot \left(1_{C(I)} \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(h)\right)$$

(2.28)
$$= \imath_*^{\mathbb{Z}/2\mathbb{Z}}(u) \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(h).$$

The proof of the second equality in (2.27) is similar.

As the last prerequisite to check that the assumptions of Proposition 2.2 are satisfied, we note the following property of the kernels of δ_i^*'s:

(2.29)
$$\delta_1^*(\text{Ker}\,\delta_2^*) = C(\mathbb{Z}/2\mathbb{Z}) \otimes \text{Ker}\,\imath_{\mathbb{Z}/2\mathbb{Z}}^*, \quad \delta_2^*(\text{Ker}\,\delta_1^*) = \text{Ker}\,\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes C(\mathbb{Z}/2\mathbb{Z}).$$

Recalling that $\sigma_i := \delta_i^* \circ \sigma$, $i = 1, 2$, we can combine $\text{Ker}\,\sigma_i = \sigma^{-1}(\text{Ker}\,\delta_i^*)$, $i = 1, 2$, with (2.29) to obtain

(2.30)
$$\sigma_1(\text{Ker}\,\sigma_2) = C(\mathbb{Z}/2\mathbb{Z}) \otimes \text{Ker}\,\imath_{\mathbb{Z}/2\mathbb{Z}}^*, \quad \sigma_2(\text{Ker}\,\sigma_1) = \text{Ker}\,\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes C(\mathbb{Z}/2\mathbb{Z}).$$

Let us now instantiate Condition (1) of Proposition 2.2 for $N = 2$:

(2.31)
$$\pi_1^0(\text{Ker}\,\pi_2^0) = \pi_0^1(\text{Ker}\,\pi_2^1), \quad \pi_2^0(\text{Ker}\,\pi_1^0) = \pi_0^2(\text{Ker}\,\pi_1^2), \quad \pi_2^1(\text{Ker}\,\pi_0^1) = \pi_1^2(\text{Ker}\,\pi_0^2),$$

where

$$\pi_1^0 := \sigma_1 \otimes \text{id}, \quad \pi_0^1 := \Phi_{01} \circ (\sigma_1 \otimes \text{id}), \quad \pi_2^0 := \sigma_2 \otimes \text{id}, \quad \pi_0^2 := \Phi_{02} \circ (\sigma_1 \otimes \text{id}),$$

(2.32)
$$\pi_2^1 := \sigma_2 \otimes \text{id}, \quad \pi_1^2 := \Phi_{12} \circ (\sigma_2 \otimes \text{id}),$$

and Φ_{ij}'s are given by (2.16). Taking advantage of (2.30), we check the first equality of (2.31):

$$
\begin{aligned}
\pi_0^1(\operatorname{Ker}\pi_2^1) &= \Phi_{01}\big((\sigma_1 \otimes \operatorname{id})(\operatorname{Ker}(\sigma_2 \otimes \operatorname{id}))\big) \\
&= \Phi_{01}\big(\sigma_1(\operatorname{Ker}\sigma_2) \otimes C(\mathbb{Z}/2\mathbb{Z})\big) \\
&= \Phi_{01}\big(C(\mathbb{Z}/2\mathbb{Z}) \otimes \operatorname{Ker}\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes C(\mathbb{Z}/2\mathbb{Z})\big) \\
&= C(\mathbb{Z}/2\mathbb{Z}) \otimes \operatorname{Ker}\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes C(\mathbb{Z}/2\mathbb{Z}) \\
&= \sigma_1(\operatorname{Ker}\sigma_2) \otimes C(\mathbb{Z}/2\mathbb{Z}) \\
&= (\sigma_1 \otimes \operatorname{id})(\operatorname{Ker}(\sigma_2 \otimes \operatorname{id})) \\
(2.33) \qquad &= \pi_1^0(\operatorname{Ker}\pi_2^0).
\end{aligned}
$$

Observe that the remaining equalities of (2.31) can be verified in the same way.

Condition (2) of Proposition 2.2 for $N = 2$ gives us 6 equalities of the form $\varphi_j^{ik} = \varphi_k^{ij} \circ \varphi_i^{jk}$, where $\varphi_k^{ij} := (\pi_k^{ij})^{-1} \circ \pi_k^{ji}$. Since $(\varphi_k^{ij})^{-1} = \varphi_k^{ji}$, these 6 equalities are pairwise equivalent. Thus it suffices to show only one of them. We choose the equality $\varphi_1^{02} = \varphi_2^{01} \circ \varphi_0^{12}$ and write it as

$$(2.34) \qquad \pi_2^{01} \circ (\pi_1^{02})^{-1} \circ \pi_1^{20} = \pi_2^{10} \circ (\pi_0^{12})^{-1} \circ \pi_0^{21}.$$

Next, denote by

$$(2.35) \qquad [\cdot]_{jk}^i : B_i \to B_i/(\operatorname{Ker}\pi_j^i + \operatorname{Ker}\pi_k^i), \quad [\cdot]_k^{ij} : B_{ij} \to B_{ij}/\pi_j^i(\operatorname{Ker}\pi_k^i),$$

the natural epimorphisms. Using the splittings of δ_i^*'s defined by (2.24) and remembering (2.32), for any $h \otimes g \otimes g' \in C(I) \otimes C(\mathbb{Z}/2\mathbb{Z}) \otimes C(\mathbb{Z}/2\mathbb{Z})$ we determine the formulae:

$$
\begin{aligned}
(\pi_1^{02})^{-1}\big([h \otimes g \otimes g']_1^{02}\big) &= [\sigma^{-1}(\hat{\omega}_2(h \otimes g)) \otimes g']_{21}^0, \\
(2.36) \qquad (\pi_0^{12})^{-1}\big([h \otimes g \otimes g']_0^{12}\big) &= [\sigma^{-1}(\hat{\omega}_2(h \otimes g)) \otimes g']_{20}^1.
\end{aligned}
$$

Furthermore, taking ω to be a linear splitting of $\sigma : \mathcal{T} \to C(S^1)$, using the notation

$$(2.37) \qquad \sigma_1(b) = 1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes b_{10} + u \otimes b_{11}, \quad \sigma_2(b) = b_{20} \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + b_{21} \otimes u, \quad b \in \mathcal{T},$$

and employing (2.32), (2.36), (2.27), for any $b \otimes g \in \mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z})$ we compute:

$$
\begin{aligned}
&(\pi_2^{01} \circ (\pi_1^{02})^{-1} \circ \pi_1^{20})([b \otimes g]_{01}^2) \\
&\quad = [\big((\sigma_1 \otimes \operatorname{id}) \circ ((\omega \circ \hat{\omega}_2) \otimes \operatorname{id}) \circ \Phi_{02} \circ (\sigma_1 \otimes \operatorname{id})\big)(b \otimes g)]_2^{01} \\
&\quad = [\big(((\delta_1^* \circ \hat{\omega}_2) \otimes \operatorname{id}) \circ \Phi_{02}\big)(1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes b_{10} \otimes g + u \otimes b_{11} \otimes g)]_2^{01} \\
&\quad = [\big((\delta_1^* \circ \hat{\omega}_2) \otimes \operatorname{id}\big)(b_{10} \otimes g \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + b_{11} \otimes g \otimes u)]_2^{01} \\
&\quad = [\imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{10}) \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{11}) \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes u]_2^{01}, \\[6pt]
&(\pi_2^{10} \circ (\pi_0^{12})^{-1} \circ \pi_0^{21})([b \otimes g]_{01}^2) \\
&\quad = [\big(\Phi_{01} \circ (\sigma_1 \otimes \operatorname{id}) \circ ((\omega \circ \hat{\omega}_2) \otimes \operatorname{id}) \circ \Phi_{12} \circ (\sigma_2 \otimes \operatorname{id})\big)(b \otimes g)]_2^{01} \\
&\quad = [\big(\Phi_{01} \circ ((\delta_1^* \circ \hat{\omega}_2) \otimes \operatorname{id}) \circ \Phi_{12}\big) \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes g + b_{21} \otimes u \otimes g)]_2^{01} \\
&\quad = [\big(\Phi_{01} \circ ((\delta_1^* \circ \hat{\omega}_2) \otimes \operatorname{id})\big)(b_{20} \otimes g \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + b_{21} \otimes g \otimes u)]_2^{01} \\
&\quad = [\Phi_{01}(\imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{20}) \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{21}) \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes u)]_2^{01} \\
(2.38) \qquad &\quad = [1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{20}) + u \otimes \imath_*^{\mathbb{Z}/2\mathbb{Z}}(g) \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{21})]_2^{01}.
\end{aligned}
$$

Hence (2.34) is satisfied provided that,

$$(2.39) \quad \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{10}) \otimes 1_{C(\mathbb{Z}/2\mathbb{Z})} + \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{11}) \otimes u = 1_{C(\mathbb{Z}/2\mathbb{Z})} \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{20}) + u \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*(b_{21}), \quad \forall\, b \in \mathcal{T}.$$

Remembering (2.37) and applying the flip to the above equation, one sees that it is equivalent to $(\mathrm{id} \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*) \circ \sigma_1 = (\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes \mathrm{id}) \circ \sigma_2$. Due to the surjectivity of σ, the latter is tantamount to $(\mathrm{id} \otimes \imath_{\mathbb{Z}/2\mathbb{Z}}^*) \circ \delta_1^* = (\imath_{\mathbb{Z}/2\mathbb{Z}}^* \otimes \mathrm{id}) \circ \delta_2^*$, which can be immediately verified.

Thus we have proven that, by Proposition 2.2, all maps $C(S_{\mathbb{R}\mathcal{T}}^2) \to (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))_i$ are surjective. Furthermore, they are by construction $\mathbb{Z}/2\mathbb{Z}$-equivariant for the diagonal action on $\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z})$. The $\mathbb{Z}/2\mathbb{Z}$-equivariance is equivalent to the $C(\mathbb{Z}/2\mathbb{Z})$-colinearity for the induced coactions. Using the gauge conjugation by (2.10), we see that $\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z})$ with the induced diagonal $C(\mathbb{Z}/2\mathbb{Z})$-coaction is a trivial principal comodule algebra. Combining all this with the fact that the kernels of the maps $C(S_{\mathbb{R}\mathcal{T}}^2) \to (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))_i$ intersect to zero, we take advantage of [15, Theorem 3.3] to conclude:

PROPOSITION 2.3. $C(S_{\mathbb{R}\mathcal{T}}^2)$ *is a principal* $C(\mathbb{Z}/2\mathbb{Z})$*-comodule algebra.*

2.4. **The tautological line bundle.** The tautological line bundle over $\mathbb{R}P^2$ can be defined as the line bundle associated with the $\mathbb{Z}/2\mathbb{Z}$-principal bundle $S^2 \to \mathbb{R}P^2$ via the antipodal action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}$. This antipodal action translates to the coaction given by the formula $1 \mapsto u \otimes 1$, where $u \in C(\mathbb{Z}/2\mathbb{Z})$ is defined by $u(\pm 1) = \pm 1$. We can now use this coaction to associate with the principal $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra $C(S_{\mathbb{R}\mathcal{T}}^2)$ a finitely generated projective left $C(\mathbb{R}P_{\mathcal{T}}^2)$-module $L := C(S_{\mathbb{R}\mathcal{T}}^2) \square_{C(\mathbb{Z}/2\mathbb{Z})} \mathbb{C}$. This is the module of the noncommutative tautological line bundle over the quantum projective space $\mathbb{R}P_{\mathcal{T}}^2$. Our primary goal is to prove that this bundle is not stably trivial, i.e., that L is not stably free.

In order to determine the K_0-class of L, we need refer to yet another isomorphic construction of $C(\mathbb{R}P_{\mathcal{T}}^2)$. Let us recall that $\mathbb{R}P^2 := S^2/(\mathbb{Z}/2\mathbb{Z})$ is homeomorphic with a disc whose boundary circle is divided by the antipodal $\mathbb{Z}/2\mathbb{Z}$-action. In the same spirit, we will show that $C(\mathbb{R}P_{\mathcal{T}}^2)$ and L are respectively isomorphic with $C(D_{\mathcal{T}})^+$ and $C(D_{\mathcal{T}})^-$ of (2.50). First we define $C(D_{\mathcal{T}})$ that will play the role of the C^*-algebra of a disk in the above construction:

(2.40)

$$C(D_{\mathcal{T}}) := \{(p_0, p_1, p_2) \in \mathcal{T}^3 \,|\, \sigma_1(p_0)(-1, x) = \sigma_1(p_1)(-1, x),$$
$$\sigma_2(p_0)(x, -1) = \sigma_1(p_2)(-1, x),$$
(2.41)
$$\sigma_2(p_1)(x, -1) = \sigma_2(p_2)(x, -1)\}.$$

Throughout this section we will frequently consider a $\mathbb{Z}/2\mathbb{Z}$-action on an algebra $C(\#)$:

$$(2.42) \qquad \alpha^\#\colon \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Aut}(C(\#)), \qquad \alpha^\#_{-1} := \alpha^\#(-1).$$

In particular, $\alpha^{\mathbb{Z}/2\mathbb{Z}}_{-1}$ is simply the pullback of the multiplication by -1. With the help of this notation, we define

$$(2.43) \qquad C(S^2_{\mathbb{R}\mathcal{T}})^\pm := \{(p_i \otimes t_i)_i \in C(S^2_{\mathbb{R}\mathcal{T}}) \mid \alpha^{\mathcal{T}}_{-1}(p_i) \otimes \alpha^{\mathbb{Z}/2\mathbb{Z}}_{-1}(t_i) = \pm p_i \otimes t_i \text{ for } i = 0,1,2\}.$$

Note that $C(S^2_{\mathbb{R}\mathcal{T}})^+ = C(S^2_{\mathbb{R}\mathcal{T}})^{\mathrm{co}\,C(\mathbb{Z}/2\mathbb{Z})}$ and L is naturally isomorphic with $C(S^2_{\mathbb{R}\mathcal{T}})^-$ (by ommitting $\otimes 1$). Thus $+$ and $-$ stand for the $\mathbb{Z}/2\mathbb{Z}$-invariant and $\mathbb{Z}/2\mathbb{Z}$-equivariant part respectively.

Next, we shall argue that $C(S^2_{\mathbb{R}\mathcal{T}})$ can be identified with the pullback C^*-algebra of the following diagram

$$(2.44)$$

$$C(S^2_{\mathbb{R}\mathcal{T}}) = C(S^2_{\mathbb{R}\mathcal{T}})^+ \oplus C(S^2_{\mathbb{R}\mathcal{T}})^-$$

with maps π_1, π_2 to $C(D_\mathcal{T})$ and $C(D_\mathcal{T})$, and maps σ°_1, σ°_2 to $C(S^1)$.

In this diagram the top maps are defined as

$$(2.45) \qquad \pi_n\colon C(S^2_{\mathbb{R}\mathcal{T}}) \ni (p_i \otimes t_i) \longmapsto (\alpha^{\mathcal{T}}_{(-1)^n}(p_i) t_i((-1)^{n+1})) \in C(D_\mathcal{T}) \text{ for } n = 1, 2.$$

To specify the maps σ°_n, first we identify six continuous functions on intervals that agree on appropriate endpoints with a continuous function on a circle. One sees that the antipodal action on S^1 pullbacks to

$$(2.46) \qquad \alpha^{S^1}_{-1}\colon C(S^1) \ni (f_1, \ldots, f_6) \longmapsto (f_4, f_5, f_6, f_1, f_2, f_3) \in C(S^1).$$

This map reflects the difference between the way in which the left $D_\mathcal{T}$ and the right $D_\mathcal{T}$ are embedded in $S^2_{\mathbb{R}\mathcal{T}}$. Now we can define $\sigma^\circ_2 := \alpha^{S^1}_{-1} \circ \sigma^\circ_1$ and

$$(2.47) \qquad \sigma^\circ_1(p_0, p_1, p_2) := \big(((\mathrm{ev}_1 \otimes \mathrm{id}) \circ \sigma_1)(p_0), ((\alpha^I_{-1} \otimes \mathrm{ev}_1) \circ \sigma_2)(p_0),$$
$$((\mathrm{id} \otimes \mathrm{ev}_1) \circ \sigma_2)(p_1), ((\mathrm{ev}_1 \otimes \alpha^I_{-1}) \circ \sigma_1)(p_1),$$
$$((\mathrm{ev}_1 \otimes \mathrm{id}) \circ \sigma_1)(p_2), ((\alpha^I_{-1} \otimes \mathrm{ev}_1) \circ \sigma_2)(p_2)\big).$$

These definitions ensure the commutativity of the diagram (2.44), i.e., $\sigma^\circ_1 \circ \pi_1 = \sigma^\circ_2 \circ \pi_2$. Hence we have a $*$-homomorphism

$$(2.48) \qquad C(S^2_{\mathbb{R}\mathcal{T}}) \ni x \longmapsto (\pi_1(x), \pi_2(x)) \in \text{ the pullback } C^*\text{-algebra of } \sigma^\circ_1 \text{ and } \sigma^\circ_2.$$

It is straightforward to verify that the above map is bijective, so that $C(S^2_{\mathbb{R}\mathcal{T}})$ is isomorphic with the pullback C^*-algebra of the diagram (2.44).

It is easily checked that the compositions $\sigma^\circ_n \circ \pi_n$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant with respect to the antipodal actions on $C(S^2_{\mathbb{R}\mathcal{T}})$ and $C(S^1)$. Indeed, on the left part of the following picture

(see (2.19) for notation) the antipodal $\mathbb{Z}/2\mathbb{Z}$-action on $C(S^2_{\mathbb{R}\mathcal{T}})$ restricted to $\sigma^\circ_1(C(D_\mathcal{T}))$ coincides with the above defined antipodal $\mathbb{Z}/2\mathbb{Z}$-action on $C(S^1)$. (See the right figure below.)
(2.49)
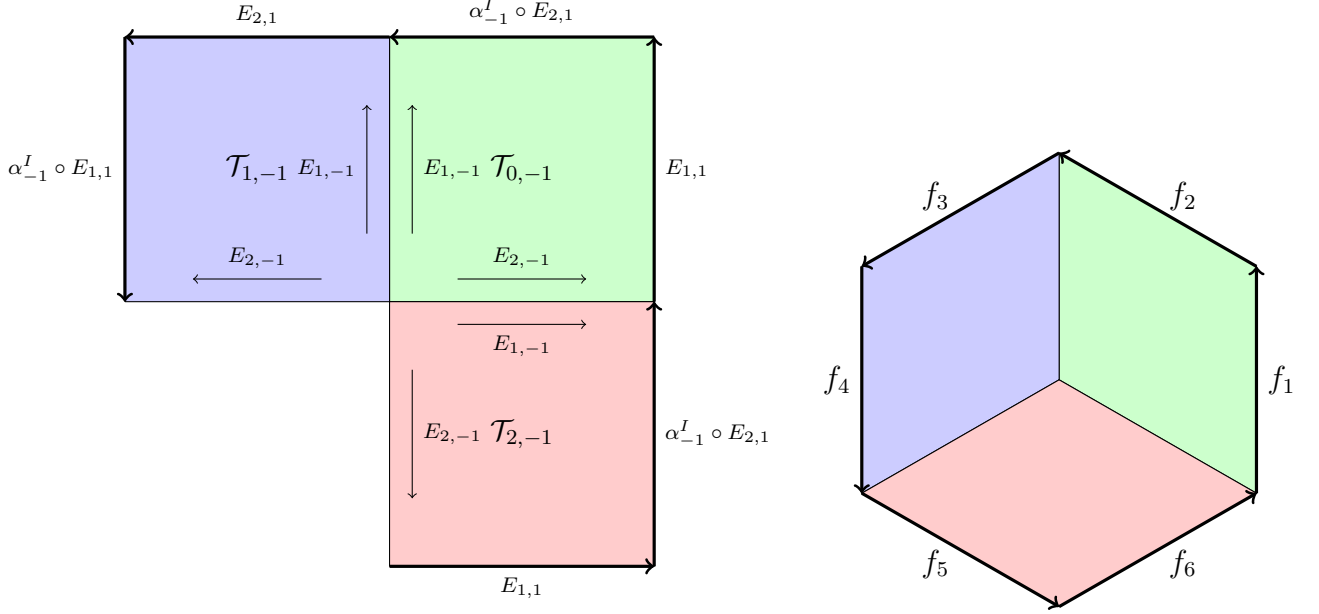
Since $L \cong C(S^2_{\mathbb{R}\mathcal{T}})^-$, our next step is to transform $C(S^2_{\mathbb{R}\mathcal{T}})^-$ to a more managable form. To this end, using (2.47) and the line above it, we define

$$(2.50) \qquad C(D_\mathcal{T})^\pm := \{(p_0, p_1, p_2) \in C(D_\mathcal{T}) \mid \sigma^\circ_2(p_1, p_2, p_3) = \pm\sigma^\circ_1(p_1, p_2, p_3)\}.$$

Next, we note that it follows from the $\mathbb{Z}/2\mathbb{Z}$-equivariance of $\sigma^\circ_n \circ \pi_n$ that $\pi^\pm_n(C(S^2_{\mathbb{R}\mathcal{T}})^\pm) \subseteq C(D_\mathcal{T})^\pm$, so that the restrictions of the $*$-homomorphisms (2.45) define

$$(2.51) \qquad \pi^\pm_n : C(S^2_{\mathbb{R}\mathcal{T}})^\pm \longrightarrow C(D_\mathcal{T})^\pm, n \in \{1, 2\}.$$

LEMMA 2.4. *Let $n \in \{1, 2\}$. The restrictions π^+_n are isomorphisms of C^*-algebras, and π^-_n are isomorphisms of modules over $C(S^2_{\mathbb{R}\mathcal{T}})^+$.*

Proof. We consider only the case $n = 1$ as the case $n = 2$ is analogous. Let us define

$$(\pi^\pm_1)^{-1} : C(D_\mathcal{T})^\pm \ni (p_0, p_1, p_2) \longmapsto (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2) \in C(S^2_{\mathbb{R}\mathcal{T}})^\pm,$$
$$(2.52) \qquad \mathbf{p}_i := \alpha^\mathcal{T}_{-1}(p_i) \otimes \mathbf{1}_1 \pm p_i \otimes \mathbf{1}_{-1},$$

where $\mathbf{1}_x$ is a function taking 1 at x and 0 everywhere else. To show that the ranges of $(\pi^\pm_1)^{-1}$ are indeed $C(S^2_{\mathbb{R}\mathcal{T}})^\pm$ respectively, first we need to check that $(\pi^\pm_1)^{-1}(C(D_\mathcal{T})^\pm) \subseteq C(S^2_{\mathbb{R}\mathcal{T}})$. To verify this inclusion, we have to check that the defining equalities (2.17) hold. We will do this only for the first equality

$$(2.53) \qquad (\sigma_1 \otimes \mathrm{id})(\alpha^\mathcal{T}_{-1}(p_0) \otimes \mathbf{1}_1 \pm p_0 \otimes \mathbf{1}_{-1}) = \big(\Phi_{01} \circ (\sigma_1 \otimes \mathrm{id})\big)(\alpha^\mathcal{T}_{-1}(p_1) \otimes \mathbf{1}_1 \pm p_1 \otimes \mathbf{1}_{-1})$$

as the remaing ones are similar. If $(p_0, p_1, p_2) \in C(D_\mathcal{T})^\pm$, it follows from (2.41) and (2.50) that

$$(2.54) \qquad \big((\mathrm{ev}_{-1} \otimes \mathrm{id}) \circ \sigma_1\big)(p_0) = \big((\mathrm{ev}_{-1} \otimes \mathrm{id}) \circ \sigma_1\big)(p_1),$$
$$\big((\mathrm{ev}_1 \otimes \mathrm{id}) \circ \sigma_1\big)(p_0) = \pm\big((\mathrm{ev}_1 \otimes \alpha^I_{-1}) \circ \sigma_1\big)(p_1).$$

Next, let us introduce the following Heynemann-Sweedler-type notation with the summation sign suppressed:

$$(2.55) \qquad \sigma_1(p) = \sigma_1(p)^{(1)} \otimes \sigma_1(p)^{(0)}, \quad \sigma_2(p) = \sigma_2(p)^{(0)} \otimes \sigma_1(p)^{(1)}.$$

Now, remembering the $\mathbb{Z}/2\mathbb{Z}$-equivariance of σ_1 and σ_2, we transform (2.53) into the following equivalent form:

$$
\begin{aligned}
(2.56) \qquad & \alpha_{-1}^{\mathbb{Z}/2\mathbb{Z}}(\sigma_1(p_0)^{(1)}) \otimes \alpha_{-1}^{I}(\sigma_1(p_0)^{(0)}) \otimes \mathbf{1}_1 \ \pm\ \sigma_1(p_0)^{(1)} \otimes \sigma_1(p_0)^{(0)} \otimes \mathbf{1}_{-1} \\
& = \mathbf{1}_1 \otimes \alpha_{-1}^{I}(\sigma_1(p_1)^{(0)}) \otimes \alpha_{-1}^{\mathbb{Z}/2\mathbb{Z}}(\sigma_1(p_1)^{(1)}) \ \pm\ \mathbf{1}_{-1} \otimes \sigma_1(p_1)^{(0)} \otimes \sigma_1(p_1)^{(1)}.
\end{aligned}
$$

One can directly check that this formula holds by evaluating the outside legs on the elements of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and using (2.54). Finally, the fact that $C(D_{\mathcal{T}})^{\pm}$ are mapped respectively to $C(S^2_{\mathbb{R}\mathcal{T}})^{\pm}$ follows immediately from the definition of $C(S^2_{\mathbb{R}\mathcal{T}})^{\pm}$ (see (2.17)).

We are ready now to verify that both compositions $\pi_1^{\pm} \circ (\pi_1^{\pm})^{-1}$ and $(\pi_1^{\pm})^{-1} \circ \pi_1^{\pm}$ are equal to identity. First, for each component of $C(D_{\mathcal{T}})^{\pm}$ we check that

$$(2.57) \qquad \pi_1^{\pm}\big(\alpha_{-1}^{\mathcal{T}}(p) \otimes \mathbf{1}_1 \pm p \otimes \mathbf{1}_{-1}\big) = \alpha_{-1}^{\mathcal{T}}(\alpha_{-1}^{\mathcal{T}}(p))\,\mathbf{1}_1(1) = p.$$

Hence $\pi_1^{\pm} \circ (\pi_1^{\pm})^{-1} = \mathrm{id}$. To see the other identity we compute

$$
\begin{aligned}
(\pi_1^{\pm})^{-1}\big(\alpha_{-1}^{\mathcal{T}}(p)t(1)\big) &= \alpha_{-1}^{\mathcal{T}}(\alpha_{-1}^{\mathcal{T}}(p))t(1) \otimes \mathbf{1}_1 \pm \alpha_{-1}^{\mathcal{T}}(p)t(1) \otimes \mathbf{1}_{-1} \\
&= p\,t(1) \otimes \mathbf{1}_1 \pm p\,(\pm\alpha_{-1}^{\mathbb{Z}/2\mathbb{Z}}(t))(1) \otimes \mathbf{1}_{-1} \\
&= p \otimes \big(t(1)\mathbf{1}_1 + t(-1)\mathbf{1}_{-1}\big) \\
(2.58) \qquad &= p \otimes t.
\end{aligned}
$$

Here to pass from the first to the second line we used the fact that

$$(2.59) \qquad \alpha_{-1}^{\mathcal{T}}(p) \otimes \alpha_{-1}^{\mathbb{Z}/2\mathbb{Z}}(t) = \pm p \otimes t \quad \Longrightarrow \quad \alpha_{-1}^{\mathcal{T}}(p) \otimes t = \pm p \otimes \alpha_{-1}^{\mathbb{Z}/2\mathbb{Z}}(t).$$

To end with, observe that π_1^{-} is an isomorphisms of modules in the sense that $\pi_1^{-}(av) = \pi_1^{+}(a)\pi_1^{-}(v)$. $\qquad \square$

To prove that $L \cong C(D_{\mathcal{T}})^{-}$ is not stably free, we will proceed along the lines of [2], where it was crucial to use the fact that $K_1(\mathcal{T}) = 0$. Here it is $D_{\mathcal{T}}$ that plays the role of $\mathcal{T}$.

LEMMA 2.5. $K_0(C(D_{\mathcal{T}})) \cong \mathbb{Z}, \qquad K_1(C(D_{\mathcal{T}})) \cong 0.$

Proof. In Section 2.3 we have proven that all maps $C(S^2_{\mathbb{R}\mathcal{T}}) \to (\mathcal{T} \otimes C(\mathbb{Z}/2\mathbb{Z}))_i$ are surjective. Combining this with (2.45) one can easily conclude that all restrictions to $C(D_{\mathcal{T}})$ of the canonical surjections are also surjective. Therefore, we can use [5, Lemma 1.8 and Theorem 1.9] to

convert the defining triple-pullback diagram (2.40) to the iterated pullback diagram:

(2.60)
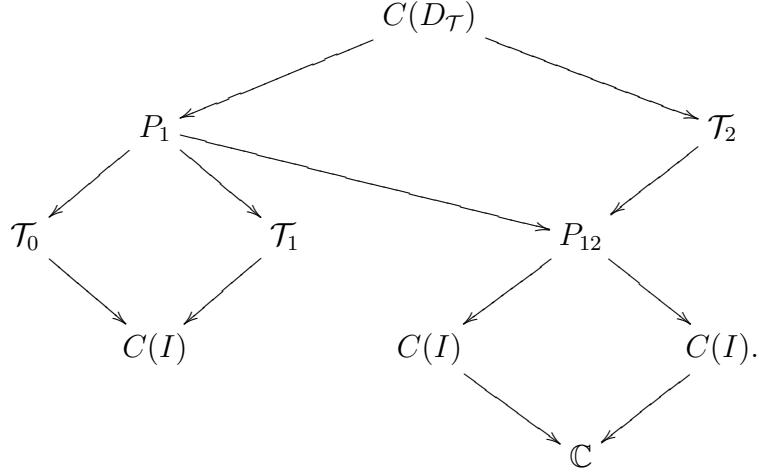
Here I is identified with an arc of S^1 as previously done (see (2.1)). Next, applying the Mayer-Vietoris six-term exact sequence to the bottom pullback sub-diagrams of the above diagram, we obtain

(2.61)

$$
\begin{array}{ccccc}
K_0(P_1) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow 0 \longleftarrow & K_1(P_1),
\end{array}
\qquad
\begin{array}{ccccc}
K_0(P_{12}) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow 0 \longleftarrow & K_1(P_{12}).
\end{array}
$$

Since $K_0(\mathcal{T}) \cong \mathbb{Z} \cong K_0(C(I))$ are generated by the classes of respective 1's in the algebras, both arrows $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ are given by the formula $(a,b) \mapsto a - b$. Hence we obtain

$$
\begin{aligned}
K_0(P_1) &= \mathbb{Z}, \quad K_0(P_{12}) = \mathbb{Z}, \\
K_1(P_1) &= 0, \quad K_1(P_{12}) = 0.
\end{aligned}
$$

This in turn yields the following form of the Mayer-Vietoris six-term exact sequence of the top pullback sub-diagram of (2.60)

(2.62)

$$
\begin{array}{ccccc}
K_0(C(D_\mathcal{T})) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow 0 \longleftarrow & K_1(C(D_\mathcal{T})).
\end{array}
$$

Finally, as the arrow $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is again (and for much the same reasons) given by the formula $(a,b) \mapsto a - b$, we conclude the claim of the lemma. $\qquad \square$

THEOREM 2.6. *Let $L := C(S^2_{\mathbb{R}\mathcal{T}}) \square_{C(\mathbb{Z}/2\mathbb{Z})} \mathbb{C}$ be the associated left $C(\mathbb{R}P^2_\mathcal{T})$-module for the coaction $\varrho \colon \mathbb{C} \to C(\mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{C}$, $\varrho(1) = u \otimes 1$, $u(\pm 1) = \pm 1$. Then L is* not *stably free. In other words, the tautological line bundle over $\mathbb{R}P^2_\mathcal{T}$ is not stably trivial.*

Proof. Suppose that L is stably free. Since $C(S^2_{\mathbb{R}\mathcal{T}})$ is principal, it follows from the stable triviality criterion [13] that there exists an invertible matrix $T \in M_n(C(S^2_{\mathbb{R}\mathcal{T}}))$ whose first row has entries in $L \cong C(S^2_{\mathbb{R}\mathcal{T}})^-$ and all other rows have entries in $C(\mathbb{R}P^2_\mathcal{T}) = C(S^2_{\mathbb{R}\mathcal{T}})^+$. Next, let $T_1 := (\pi_1(T_{ij}))$ (see (2.45) for π_1) be the corresponding invertible matrix over $C(D_\mathcal{T})$. Then, by Lemma 2.4, the first row of T_1 has entries in $C(D_\mathcal{T})^-$ and all other rows have entries in $C(D_\mathcal{T})^+$.

Furthermore, applying σ_1° of (2.47) componentwise to T_1, we obtain an invertible matrix T_2 over $C(S^1)$. It follows directly from the definition of $C(D_\mathcal{T})^\pm$ that the determinant of this matrix is a $\mathbb{Z}/2\mathbb{Z}$-equivariant function, i.e. $\det(T_2)(-t) = -\det(T_2)(t)$. A standard topological argument shows that the winding number of such a function (normalized to a function from S^1 to S^1) is odd. Hence the K_1 class of T_2 is odd. On the other hand, this class equals to $\sigma_{1*}^\circ([T_1]_{K_1(C(D_\mathcal{T}))})$. This contradicts the fact that $K_1(C(D_\mathcal{T})) = 0$ (see Lemma 2.5). $\qquad\square$

Consider now the obvious Hopf algebra surjection $\pi \colon \mathcal{O}(U(1)) \to C(\mathbb{Z}/2\mathbb{Z})$. This yields the prolongation $C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathcal{O}(U(1))$ (cf. [2]). Since $C(S^2_{\mathbb{R}\mathcal{T}})$ is a principal $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra, it follows from [24, Lemma 2.3] that $C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathcal{O}(U(1))$ is a principal $\mathcal{O}(U(1))$-comodule algebra. On the other hand, as $L := C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathbb{C}$ is not free due to Theorem 2.6, we conclude that the $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra $C(S^2_{\mathbb{R}\mathcal{T}})$ is not cleft. Likewise, since

$$(2.63) \qquad C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathbb{C} \cong C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathcal{O}(U(1))\square_{\mathcal{O}(U(1))}\mathbb{C},$$

we can view L as a module associated to the $\mathcal{O}(U(1))$-comodule algebra $C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathcal{O}(U(1))$. Hence the latter is also not cleft. Furthermore, since the $C(S^2_{\mathbb{R}\mathcal{T}})$ and $\widetilde{C(S^2_{\mathbb{R}\mathcal{T}})}$ are isomorphic as $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebras, and the latter is a piecewise trivial principal $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra by construction, so is $C(S^2_{\mathbb{R}\mathcal{T}})$. Combining this with Lemma 1.13 and the obvious fact that $\mathcal{O}(U(1))$ is a principal $C(\mathbb{Z}/2\mathbb{Z})$-comodule algebra, we can apply Theorem 1.5 to conclude that $C(S^2_{\mathbb{R}\mathcal{T}})\square_{C(\mathbb{Z}/2\mathbb{Z})}\mathcal{O}(U(1))$ admits a $\operatorname{Ker}\pi$-reduction. Thus we obtain a non-trivial illustration of Theorem 1.5: a non-cleft piecewise trivial and reducible principal comodule algebra.

3. The $SU_q(2)$-prolongation of the classical Hopf fibration

To fix the notation let us recall the definition of polynomial $*$-algebra $\mathcal{O}(SU_q(2))$ and Peter-Weyl algebra $\mathrm{PW}_{U(1)}(C(S^3))$ of functions on classical S^3 sphere. For details on the latter algebra we refer the reader to [3].

Recall that the algebra of polynomial functions on $SU_q(2)$ is generated as a $*$ algebra by α and γ satisfying relations

$$(3.1) \qquad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

The Hopf algebra structure is given in a usual way by a matrix co-representation

$$(3.2) \qquad U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

i.e., $\Delta(U) = U \otimes U$, $S(U) = U^\dagger$, $\varepsilon(U) = I$. The Hopf $*$-algebra epimorphism

$$(3.3) \qquad \pi : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{O}(U(1)), \quad \pi(\alpha) = u, \quad \pi(\gamma) = 0$$

makes $\mathcal{O}(SU_q(2))$ into a left and right $\mathcal{O}(U(1))$-comodule via left and right coactions $(\pi\otimes\mathrm{id})\circ\Delta$ and $(\mathrm{id}\otimes\pi)\circ\Delta$.

The comodule algebra $\mathrm{PW}_{U(1)}(C(S^3))$ ([3]) called the Peter-Weyl algebra extends the notion of the algebra of regular functions on (spanned by the matrix coefficients of the irreducible unitary corepresentations) from compact quantum groups to unital C^*-algebras on which they act. The history of this concept and its fundamental relationship to the canonical decomposition

of a compact-group representation into isotypical components is explained in [22]. The Peter-Weyl algebra is a comodule algebra over the Hopf algebra of regular functions on a compact quantum group. In general, it is not a C^*-algebra, although its coaction-invariant (the base space) subalgebra is always a C^*-algebra. Now, one can show that the Peter-Weyl comodule algebra of functions on a compact Hausdorff space with an action of a compact group is principal if and only if the action is free [1]. In other words, the Galois condition of Hopf-Galois theory holds if and only if we have a compact principal bundle.S

$$\mathcal{O}(S^3) \subsetneq \mathrm{PW}_{U(1)}(C(S^3)) \subsetneq C(S^3).$$

Let us denote by $a, c : S^3 \to \mathbb{C}$ the coordinate functions on S^3 satisfying $|a|^2 + |c|^2 = 1$. The action of $U(1)$ on S^3 dualises to the $U(1)$-comodule algebra structure on a dense subalgebra of $C(S^3)$, in particular the $\mathcal{O}(U(1))$-coaction on coordinate function is given by

$$a \longmapsto a \otimes u, \quad c \longmapsto c \otimes u.$$

The line bundles on S^3 are classified by integers. Let us denote the n-th line bundle on S^3 by L_n, $n \in \mathbb{Z}$. The left $C(S^2)$-module of sections of L_n can be written as

$$(3.4) \qquad \Gamma(L_n) = C(S^3) \square_{\mathcal{O}(U(1))}{}^n\mathbb{C} \subseteq C(S^3),$$

where ${}^n\mathbb{C}$ is $\mathbb{C}$ considered as a left $\mathcal{O}(U(1))$-comodule, with the coaction given by $k \mapsto u^n \otimes k$, $k \in \mathbb{C}$. Equivalently, denoting by $\hat{\Delta} : C(S^3) \to C(S^3 \times U(1))$ the "completed" co-multiplication induced from the standard action of $U(1)$ on S^3 we have

$$(3.5) \qquad \Gamma(L_n) = \{f \in C(S^3) \mid \hat{\Delta}(f) = f \otimes u^n\}.$$

Finally, the Peter-Weyl algebra of S^3 is defined as

$$(3.6) \qquad \mathrm{PW}_{U(1)}(C(S^3)) = \bigoplus_{n \in \mathbb{Z}} \Gamma(L_n),$$

where the direct sum is taken in the category of left $C(S^2)$-modules. Now we will describe the piecewise trivial structure of $\mathrm{PW}_{U(1)}(C(S^3))$. First let us denote for brevity

$$(3.7) \qquad \omega := \sqrt{\frac{2}{1 + ||a|^2 - |c|^2|}}.$$

Note that $\omega \in C(S^2) \subseteq \mathrm{PW}_{U(1)}(C(S^3))$. Let us denote the following two ideals $I_1, I_2 \subseteq C(S^2)$:

$$I_a = \{f \in C(S^2) \mid f(x) = 0 \text{ for all } x \in S^2 \text{ such that } |a|^2(x) \le 1/2\},$$
$$(3.8) \qquad I_c = \{f \in C(S^2) \mid f(x) = 0 \text{ for all } x \in S^2 \text{ such that } |a|^2(x) \ge 1/2\}.$$

It is well known that (c.f. [3]) $I_a \cap I_c = 0$, $C(S^2)/I_i \simeq C(D)$, $i = a, c$, where D is a unit disk, and that

$$(3.9) \qquad (1 - \omega^2|a|^2) \in I_a, \quad (1 - \omega^2|c|^2) \in I_c.$$

Note that ([3, Eq. 3.4.57])

$$(3.10) \qquad (1 - \omega^2|a|^2)(1 - \omega^2|c|^2) = 0.$$

The covering of $\mathrm{PW}_{U(1)}(C(S^3))$ can now be given by the canonical surjections in terms of I_a and I_c (cf. [3]):

$$\pi_a : \mathrm{PW}_{U(1)}(C(S^3)) \longrightarrow \mathrm{PW}_{U(1)}(C(S^3))/(I_a\mathrm{PW}_{U(1)}(C(S^3))),$$

(3.11) $$\pi_c : \mathrm{PW}_{U(1)}(C(S^3)) \longrightarrow \mathrm{PW}_{U(1)}(C(S^3))/(I_c\mathrm{PW}_{U(1)}(C(S^3))).$$

Because $\mathrm{PW}_{U(1)}(C(S^3))$ is a principal $C(U(1))$-comodule algebra with $C(S^2) = \mathrm{PW}_{U(1)}(C(S^3))^{\mathrm{co}\, C(U(1))}$ ([3]), by [15, Proposition 3.4] we have that $\mathrm{Ker}\, \pi_a \cap \mathrm{Ker}\, \pi_c = \{0\}$, so maps π_i form a covering. Hence $\mathrm{PW}_{U(1)}(C(S^3))$ is a piecewise trivial, principal ([3]) comodule algebra.

The possible trivialisation associated with the above covering is given by the following cleaving maps which are clearly algebra morphisms:

$$\gamma_a : \mathcal{O}(U(1)) \longrightarrow \mathrm{PW}_{U(1)}(C(S^3))/(I_a\mathrm{PW}_{U(1)}(C(S^3))), \qquad u^n \longmapsto \pi_a(\omega a)^n,$$

(3.12) $$\gamma_c : \mathcal{O}(U(1)) \longrightarrow \mathrm{PW}_{U(1)}(C(S^3))/(I_c\mathrm{PW}_{U(1)}(C(S^3))), \qquad u^n \longmapsto \pi_c(\omega c)^n.$$

$\mathrm{PW}_{U(1)}(C(S^3))$ itself is not cleft ([3]). In fact one can argue (cf. [3]) that

$$f_a : \mathrm{PW}_{U(1)}(C(S^3))/(I_a\mathrm{PW}_{U(1)}(C(S^3))) \xrightarrow{\simeq} C(D) \otimes \mathcal{O}(U(1)),$$

(3.13) $$f_c : \mathrm{PW}_{U(1)}(C(S^3))/(I_c\mathrm{PW}_{U(1)}(C(S^3))) \xrightarrow{\simeq} C(D) \otimes \mathcal{O}(U(1)),$$

where the identifications are induced from respective cleavings 3.12 using isomorphisms

(3.14) $$f_i : x \longmapsto \pi_i(x_{(0)})\gamma_i(Sx_{(1)}) \otimes x_{(2)}, \quad i = a, c.$$

Indeed, it is well known that $\pi_a(C(S^2)) \simeq C(D) \simeq \pi_c(C(S^2))$. Then, for any $n \in \mathbb{Z}$

(3.15) $$f_i(\pi_i(\Gamma(L_n))) = \pi_i(C(S^2)) \otimes u^n = C(D) \otimes u^n, \quad i = a, c,$$

and hence $f_i(\mathrm{PW}_{U(1)}(C(S^3))) = C(D) \otimes \mathcal{O}(U(1))$. Indeed, by the definition of f_i and $\Gamma(L_n)$ it is obvious that $f_i(\pi_i(\Gamma(L_n))) \subseteq \pi_i(C(S^2)) \otimes u^n$. On the other hand, consider an arbitrary element $y \in C(D)$. Then there exist elements $y_a, y_c \in C(S^2)$ such that $y = \pi_a(y_a) = \pi_c(y_c)$. Then

(3.16) $$y \otimes u^n = f_z(\pi_z(y_z z^n \omega^n)), \quad z = a, c,$$

where we abuse the notation slightly using the convention that $z^{-|n|} := (z^*)^{|n|}$ even for non-unitary z.

Because $\mathcal{O}(SU_q(2))$ is a principal left $\mathcal{O}(U(1))$-comodule algebra, by the Lemma 1.13 also $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$ is a principal, piecewise trivial comodule algebra with trivialisations and covering inherited from $\mathrm{PW}_{U(1)}(C(S^3))$ with the formulas:

$$\hat{\pi}_i = \pi_i \otimes \mathrm{id}, \quad \hat{\gamma}_i = (\gamma_i \circ \pi \otimes \mathrm{id}) \circ \Delta_{\mathcal{O}(SU_q(2))}, \quad i = a, c.$$

Hence, using the isomorphisms 3.13 it follows that $\mathrm{PW}_{U(1)}(C(S^3))$ can be thought of equivalently as an appropriate gluing (cf. eq. 1.8) of two copies of $C(D) \otimes \mathcal{O}(SU_q(2))$.

By the Theorem 1.5 $\mathrm{PW}_{U(1)}(C(S^3))$ is a piecewise trivial $\mathrm{Ker}\,\pi$-reduction (3.3) of a piecewise trivial comodule algebra $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$. Note, that as the comodule algebra $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$ is a cotensor product the condition on the transition functions and the trivialisations of $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$ is automatically satisfied by Lemma 1.13.

The following result states that $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$ is a non-smash product comodule algebra.

THEOREM 3.1. $\mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$ *is not isomorphic as a comodule algebra to any smash product* $C(S^2)\#\mathcal{O}(SU_q(2))$.

Proof. Suppose there exists a cleaving map

$$\mathcal{O}(SU_q(2)) \longrightarrow \mathrm{PW}_{U(1)}(C(S^3))\square_{\mathcal{O}(U(1))}\mathcal{O}(SU_q(2))$$

that is an algebra homomorphism. It is tantamount to the existence of a $U(1)$-equivariant algebra homomorphism $\mathcal{O}(SU_q(2)) \to \mathrm{PW}_{U(1)}(C(S^3))$ ([7, Proposition 4.1]). Let α and γ denote generators of $\mathcal{O}(SU_q(2))$ and a, c their classical counterparts. Since $f([\alpha, \alpha^*]) = 0$, we have $f(\gamma) = 0$. On the other hand by the $U(1)$-equivariance $f(\alpha) = f_1 a + f_2 c$, $f_1, f_2 \in C(S^2)$, and due to the sphere equation $f(\alpha)f(\alpha)^* = 1$.

Now, any continuous section of the Hopf line bundle L_1 can be written as $g_1 a + g_2 c$ for some $g_1, g_2 \in C(S^2)$. We can rewrite it as $(g_1 a + g_2 c)f(\alpha)^* f(\alpha)$. Since $(g_1 a + g_2 c)f(\alpha)^* \in C(S^2)$, we conclude that $f(\alpha)$ spans $\Gamma(L_1)$ as a left $C(S^2)$-module. Furthermore, if $gf(\alpha) = 0$ for some $g \in C(S^2)$ then $g = gf(\alpha)f(\alpha^*) = 0$. Hence $f(\alpha)$ is a basis of L_1 contradicting its non-triviality. $\square$

4. THE IRREDUCIBILITY OF A QUANTUM PLANE FRAME BUNDLE

The aim of this Section is to show that the frame bundle of the quantum plane $\mathbb{C}_q$ is not reducible to an $SL_q(2)$-sub-bundle unless q is a cubic root of 1 [17]. To this end, we will need:

PROPOSITION 4.1. *For a smash product* $P = B \rtimes H$, *the elements* $f \in \mathrm{Alg}_H^H(^{coH/J}H, Z_P(B))$ *are in bijective correspondence with unital linear maps* $\vartheta : {}^{coH/J}H \to B$ *satisfying, for all* $k, l \in {}^{coH/J}H$, $h \in H$, $b \in B$,

$$(4.1) \qquad \vartheta(kl) = \vartheta(l)\vartheta(k), \quad b\vartheta(k) = \vartheta(k_{(1)})(k_{(2)} \triangleright b), \quad \vartheta(Sh_{(1)}kh_{(2)}) = Sh \triangleright \vartheta(k).$$

The correspondence is given explicitly by

$$(4.2) \qquad\qquad f \longmapsto \vartheta_f = (\mathrm{id}_B \otimes \varepsilon) \circ f, \quad \vartheta \longmapsto f_\vartheta = (\vartheta \otimes \mathrm{id}_H) \circ \Delta.$$

Proof. The correspondence (4.2) can be proven using the right H-colinearity of f. Next, put $D := {}^{coH/J}H$. Then $bf(k) = f(k)b$ for all $k \in D$ and $b \in B$. Explicitly,

$$(4.3) \qquad\qquad bf(k) = b\vartheta(k_{(1)}) \otimes k_{(2)} \qquad \text{and} \qquad f(k)b = \vartheta(k_{(1)})(k_{(2)} \triangleright b) \otimes k_{(3)}.$$

Hence the second equality in (4.1) follows. In order to prove the first one, we use the fact that f is an algebra homomorphism. For any $k, l \in D$, we have $f(kl) = \vartheta(k_{(1)}l_{(1)}) \otimes k_{(2)}l_{(2)}$. On the other hand,

$$(4.4) \qquad f(kl) = f(k)f(l) = (\vartheta(k_{(1)}) \otimes k_{(2)})(\vartheta(l_{(1)}) \otimes l_{(2)}) = \vartheta(k_{(1)})(k_{(2)} \triangleright \vartheta(l_{(1)})) \otimes k_{(3)}l_{(2)}.$$

Therefore, the already proven second property from (4.1) and the fact that $\vartheta(l) \in B$ yield

$$(4.5) \qquad\qquad\qquad \vartheta(kl) = \vartheta(k_{(1)})(k_{(2)} \triangleright \vartheta(l)) = \vartheta(l)\vartheta(k).$$

Finally, the last property of ϑ follows from the invariance of f with respect to the Miyashita-Ulbrich H-action. We end this proof by noting that using the above arguments backwards shows that, if the map $\vartheta : D \to B$ satisfies (4.1), then the map $k \mapsto \vartheta(k_{(1)}) \otimes k_{(2)}$ belongs to $\mathrm{Alg}_H^H({}^{coH/J}H, Z_{B \rtimes H}(B))$. $\square$

We are now ready to demonstrate that $B \rtimes H$, where $B = A(\mathbb{C}_q^2)$ and $H = A(GL_q(2))$ is not reducible to an $A(SL_q(2))$-bundle, unless $q^3 = 1$. Recall that $A(\mathbb{C}_q^2)$ is defined as the unital associative algebra over $\mathbb{C}$ generated by x, y with relations

$$(4.6) \qquad\qquad xy = qyx, \quad q \in \mathbb{C} \setminus \{0\},$$

and $A(GL_q(2))$ is defined as the unital associative algebra over $\mathbb{C}$ generated by a, b, c, d, D^{-1} with relations

$$(4.7) \qquad ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad = da + (q - q^{-1})bc$$

$$(4.8) \qquad\qquad (ad - qbc)D^{-1} = D^{-1}(ad - qbc) = 1,$$

where $q \in \mathbb{C} \setminus \{0\}$. The Hopf algebra structure of $A(GL_q(2))$ is defined in terms of the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of generators in the usual way.

There exists a well-defined left action of $A(GL_q(2))$ on $A(\mathbb{C}_q^2)$ given by the formulas

$$(4.9) \qquad a \triangleright x = q^{-2}x, \ b \triangleright x = 0, \ c \triangleright x = (q^{-2} - 1)y, \ d \triangleright x = q^{-1}x, \ D^{-1} \triangleright x = q^3 x,$$

$$(4.10) \qquad a \triangleright y = q^{-1}y, \ b \triangleright y = 0, \ c \triangleright y = 0, \ d \triangleright y = q^{-2}y, \ D^{-1} \triangleright y = q^3 y.$$

Denote by $\pi : A(GL_q(2)) \to A(SL_q(2))$ the natural surjection sending D to 1. Suppose that there exists a $\mathrm{Ker}\,\pi$-reduction of $B \rtimes H$. It follows from Lemma 4.1 that there exists a unital and anti-algebra map $\vartheta : {}^{co\,A(SL_q(2))}H \to B$. In particular, as $D, D^{-1} \in {}^{co\,A(SL_q(2))}H$ and

$$(4.11) \qquad 1 = \vartheta(1) = \vartheta(DD^{-1}) = \vartheta(D^{-1})\vartheta(D) \quad \text{and} \quad 1 = \vartheta(1) = \vartheta(D^{-1}D) = \vartheta(D)\vartheta(D^{-1}),$$

we obtain that $\vartheta(D^{-1})$ is an invertible element of $B = A(\mathbb{C}_q^2)$. Since the only invertible elements of $A(\mathbb{C}_q^2)$ are multiples of identity, we conclude that $\vartheta(D^{-1}) = \mu 1_B$, with $0 \neq \mu \in \mathbb{C}$. On the other hand, from Lemma 4.1 and eq. (4.9) we obtain that

$$(4.12) \qquad\qquad \mu x = x\vartheta(D^{-1}) = \vartheta(D^{-1})(D^{-1} \triangleright x) = q^3 \mu x,$$

so that $q^3 = 1$, as claimed.

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